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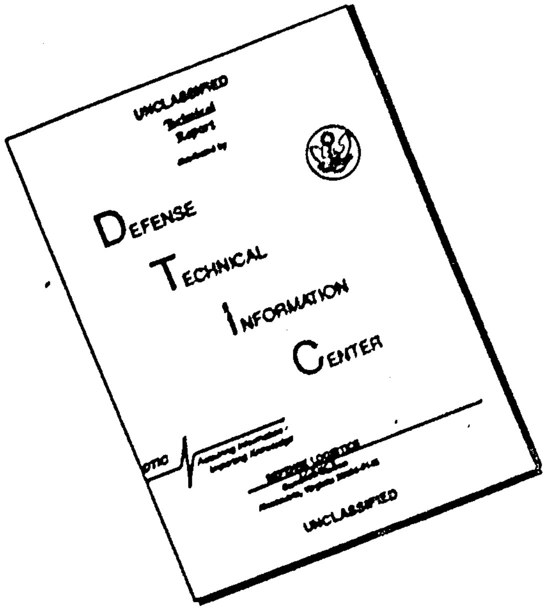

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FUNDAMENTAL RESEARCH IN APPLIED MATHEMATICS

ON THE NUMERICAL SOLUTION OF LINEAR AND NON-LINEAR PARABOLIC EQUATIONS ON THE ORDVAC

by

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On the Numerical Solution of Linear and Non-Linear Parabolic Equations on the Ordvac

by

David Young and Louis Ehrlich

Table of Contents

1. Introduction

Let us consider the problem of finding a function $u(x,y)$ which satisfies the parabolic partial differential equation

$$(1.1) \qquad u_t = u_{xx} + f(x,t,u)$$

in R: $0 < x < A$, $0 < t < T$ and which satisfies the initial conditions

$$(1.2) \qquad u(x,0) = g(x) \qquad\qquad 0 < x < A$$

and the boundary conditions

$$(1.3) \qquad \begin{cases} \alpha_1(t)u_x(0,t) + \beta_1(t)u(0,t) = \phi(t) & 0 \leq t \\ \alpha_2(t)u_x(A,t) + \beta_2(t)u(A,t) = \psi(t) & 0 \leq t . \end{cases}$$

The function $u(x,y)$ is required to be continuous and bounded in R + S, where S is the boundary of R, except that continuity of $u(x,y)$ is not required at a point of S where a jump discontinuity occurs in any of the other functions involved in (1.2) and (1.3). We assume that all functions are continuous and bounded in R.

An analytic solution for the above problem is available only under very special conditions. We shall not seek such solutions here. Moreover, we shall not even consider the very important questions of existence and uniqueness. Rather, we shall be concerned with the study of numerical procedures for solving certain difference equations which one obtains when one uses finite difference methods. Of course it is recognized that any solutions so obtained would be of no value in the event that no solution of the original problem existed or in the event that more than one solution existed. In certain cases existence

and uniqueness are available (See for instance [1] and [6]).

The numerical methods which we shall consider include the well-known Forward Difference method, the Crank-Nicolson method [4] and the DuFort-Frankel method [5]. With the Crank-Nicolson method one is usually required to solve at each step a system of simultaneous equations. This can be done effectively in many cases using the Successive Overrelaxation method of iteration [22]. Alternatively if the system is linear, or is replaced by a related linear system, one can use a non-iterative method, apparently first used for this type of problem by Bruce, Peaceman, Rachford, and Rice [2].

In section 2 we first derive finite difference representations of the boundary conditions including two well-known methods and one which we have not seen in the literature. We then derive several difference equations and obtain local error estimates using methods of numerical quadrature. Next, we consider the process known as "extrapolation to zero grid size", first used by L.F. Richardson [20]. Using this process one can, under favorable conditions, appreciably improve the accuracy of the finite difference methods. We indicate how results obtained using several different grid sizes can often be used to obtain a more accurate extrapolation formula than one would obtain using Richardson's formula wherein it is assumed that the error tends to zero like the square of the grid size.

In order to compare the various finite difference methods we prepared a comprehensive routine for Ordvac which is capable of solving a large class of problems by each method. Details of this program are given in section 4 and Appendix. The specific problems which we considered are discussed in section 3.

The program may be considered as a research tool to enable one to study the effectiveness of the various methods under different conditions. When used together with a more complete theoretical analysis than we have been able to carry out here, it is to be expected that new and significant results on finite difference methods should be obtained.

In section 5 the results obtained on the Ordvac are analysed and tentative conclusions are reached as to the order of convergence and the overall effectiveness of each of the methods. For the linear case our results indicate strongly that the use of the Crank-Nicolson method with the non-iterative procedure, is superior to any of the other methods. In the non-linear case however, one loses accuracy with this method if one modifies the difference equation in an attempt to avoid all iterations. Further work appears to be indicated in order to preserve the accuracy of the difference equation without requiring too many iterations. We suggest one possible scheme for doing this, but we have not yet tried it out on the machine.

For the Forward Difference method and for the Crank-Nicolson method, used with the iterative process, we found that the accuracy could usually be improved considerably by the use of the proper extrapolation to zero grid size. Such extrapolation was more difficult to carry out for the Crank-Nicolson method, used without iterations.

As expected the use of the proposed method for representing the boundary conditions appeared to be more accurate than either of the other two considered.

Although this study is far from complete, it was felt that the work which has been done to date should be reported now, especially since no further work can be done during the next academic year. The work may be regarded

as a natural extension of research which was begun by the Interior Ballistics Laboratory, Aberdeen Proving Ground. One of the objects was to determine the order of convergence of the Forward Difference method. Extensive calculations were carried out on the Bell Relay Computer, as described in [6], and on the Eniac. The latter calculations have not previously been reported. We shall study them in section 5 along with the Ordvac results.

In addition to acknowledging the sponsorship of the Office of Ordnance Research, U.S. Army, we should also like to express our appreciation for the encouragement and assistance furnished by the Interior Ballistic Laboratory at the Aberdeen Proving Ground. Discussions with Dr. W.C. Taylor of the Interior Ballistic Laboratory and with Dr. B.L. Hicks, formerly of that laboratory, were of great value. We are also pleased to acknowledge the assistance of Dr. R.M. Conlan, Miss J.M. Wood, Mr. B.T.C. Koo, and Mr. S. Soscia who worked at various times on the programming of the problem for Ordvac.

2. Difference Equations

Let h and k be positive numbers such that A/h is an integer, which we denote by M, and let $L_{h,k}$ denote the set of points (x,t) such that x/h and t/k are integers. Let $R_{h,k}$ and $S_{h,k}$ denote respectively the set of points of $L_{h,k}$ belonging to R and to S,(See Figure 1). We replace the problem of the previous section by that of finding a function u(x,t) defined on $R_{h,k} + S_{h,k}$, satisfying a certain difference equation on $R_{h,k}$ and satisfying on $S_{h,k}$ certain other conditions derived from (1.2) and (1.3).

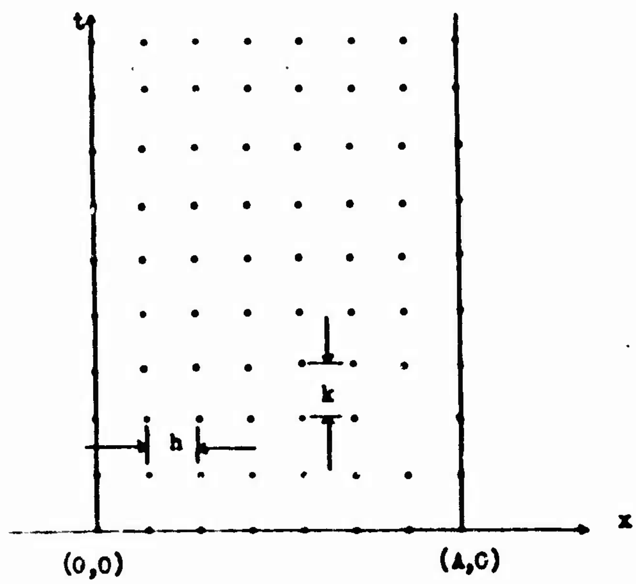


Figure 1

In this section we derive finite difference equations which are obtained from the differential equation and from the boundary conditions, and seek to estimate the accuracy. Our methods are by no means rigorous. Indeed they are merely formal methods utilizing Taylor series expansions whenever convenient. Nevertheless, it is felt that these methods will give some basis for making comparisons of the accuracies of the various finite difference methods, at least

for some cases. We will later test the comparisons thus obtained against numerical results obtained for typical problems on the Ordvac.

For discussion of finite difference analogues of problems involving the infinite region $-\infty < x < \infty, t > 0$ the reader is referred to papers by Courant, Friedrichs and Lewy [3] and by Fits John [10].

In addition to deriving the difference equations we will also describe here the associated numerical procedures. In order that we may be able to give a complete description of these procedures it is convenient to first consider the treatment of the boundary conditions.

2.1 Boundary conditions. Although our discussion could be extended to cover more general situations, we shall consider only the following two types of boundary conditions

$$(2.1.1)\qquad \begin{cases} \text{(a)}\ \ u(0,t) = \phi(t), & (t \geq 0) \\ \text{(b)}\ \ u(A,t) = \psi(t), & (t \geq 0), \end{cases}$$

and

$$(2.1.2)\qquad \begin{cases} \text{(a)}\ \ u_x(0,t) = -H(u_g - u(0,t)), & (t > 0) \\ \text{(b)}\ \ u_x(A,t) = 0, & (t > 0). \end{cases}$$

Here H and u_g are constants.

We shall at times refer to the boundary conditions represented by (2.1.1) and (2.1.2) as B V and N D conditions, respectively.

We first note that the initial conditions (1.2) are easily represented by

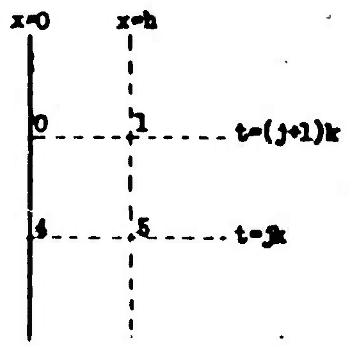
$$(2.1.3)\qquad u_{i,0} = g_i \qquad 1 \leq i \leq M-1 .$$

Here and subsequently we let $u_{i,j}=u(ih, jk)$, $g_i=g(ix)$,etc., where i and j are integers. Moreover, the B V conditions (2.1.1) can be replaced by

$$(2.1.4)\qquad \begin{cases} u_{0,j} = \phi_j, & (0 \leq j) \\ u_{M,j} = \psi_j, & (0 \leq j). \end{cases}$$

However, with the ND condition (2.1.2a) we shall consider several alternative procedures. A number of these are considered by Price and Slack [9]. We shall discuss two of these together with one which may very well be known also, although we have not seen it in the literature.

In order to simplify our notation we shall consider a configuration of four points near the line x = 0, as shown in Figure 2. We denote by u_i, t_i, x_i, f_i the values of the respective variables at the point numbered i, (i = 0, 1, 4, 5). We seek relations between u_0 and one or more of the other u_i. It



Typical Boundary Configuration - Left Hand Boundary

Figure 2

is assumed that u_4 and u_5 have already been obtained by previous calculations.

Perhaps the most obvious method is to replace $u_x(0,t)$ by $(u_1 - u_0)h^{-1}$ in (2.1.2a) obtaining

(2.1.5) $$u_0 \sim \frac{u_1 + hHu_E}{1 + Hh} \quad .$$

Formula (2.1.5) defines a procedure for representing (2.1.2a) which we shall refer to as <u>Option I</u>.

To study the accuracy of (2.1.5) we expand $u(x,t)$ in a Taylor series about the point (x_0, t_0) obtaining

$$u_1 = u_0 + hu_x + \tfrac{1}{2}h^2 u_{xx} + \ldots \quad .$$

Using (2.1.2a) we obtain

(2.1.6) $$u_0 - \frac{u_1 - hHu_E}{1 + Hh} = -\frac{h^2}{2(1+Hh)} u_{xx} + \ldots \quad .$$

We note that the leading term of the local error, given by the right member of (2.1.6) is $O(h^2)$. We shall not try to estimate here the effect of local errors on the accuracy of the solution of the difference equation as a solution of the

differential equation. Instead we shall attempt to infer this from the results of Ordvac computations.

If one makes use of the differential equation at the point (x_4,t_4), one can obtain more accurate formulas. Thus, using a Taylor series expansion about (x_4,t_4) we have

$$u_0 = u_4 + ku_t + \tfrac{1}{2}k^2u_{tt} + \ldots$$

$$u_5 = u_4 + hu_x + \tfrac{1}{2}h^2u_{xx} + \tfrac{1}{6}h^3 u_{xxx} + \ldots \quad .$$

Here the derivatives are evaluated at (x_4,t_4). Using (1.1) and the relation $u_x = -H(u_g - u_4)$ we obtain

$$u_0 \sim u_4(1-2r) + 2ru_5 + 2rhH(u_g - u_4) + kf_4 \tag{2.1.7}$$

where the leading terms of the difference between the right and the left members of (2.1.7), that is, the right member minus the left member, are

$$-\frac{kh}{3}u_{xxx} + \frac{k^2}{2}u_{tt} \; . \tag{2.1.8}$$

Here and subsequently we let

$$r = \frac{k}{h^2} \; . \tag{2.1.9}$$

Price and Slack [19] show that this formula introduces instability for $r > \frac{1}{2}$, in contrast to (2.1.5) which is stable for all r. Thus assuming $k = rh^2$, where r is a constant not greater than $\frac{1}{2}$, we obtain from (2.1.8)

$$-\frac{rh^3}{3}u_{xxx} + \frac{r^2h^4}{2}u_{tt} \; . \tag{2.1.10}$$

Thus the leading term of the local error is $O(h^3)$ as compared with $O(h^2)$ with Option I. Unfortunately, however, as we shall see later, the limitation which must be imposed on k is very severe.

We shall refer to (2.1.7) as Option II.

We now seek a formula which is as accurate as Option II and which is, at the same time, stable for all r. With this aim we expand u(x,t) in a Taylor series about the point (x_0,t_0) obtaining

$$u_1 = u_0 + hu_x + \tfrac{1}{2}h^2u_{xx} + \frac{1}{6}h^3u_{xxx} + \ldots$$

$$u_4 = u_0 - ku_t + \tfrac{1}{2}k^2u_{tt} - \frac{1}{6}k^3u_{ttt} + \ldots \quad .$$

Using (2.1.2a) and (1.1) we have

$$u_0 \backsim \frac{u_1 + hHu_g + \frac{1}{2r}u_4 + \frac{1}{2}h^2f_0}{1 + hH + \frac{1}{2r}} \tag{2.1.11}$$

where the leading term of the difference between the right and left members of (2.1.11) is

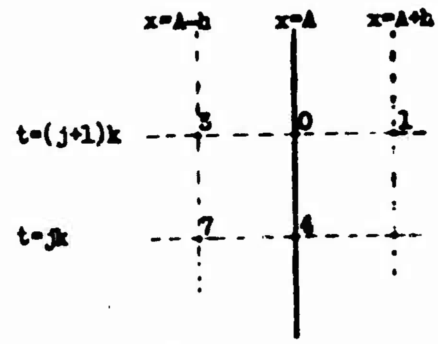
$$\frac{-\frac{1}{4}h^2ku_{tt} - \frac{1}{6}h^3u_{xxx}}{1 + hH + \frac{1}{2r}} \quad . \tag{2.1.12}$$

Evidently the leading term of the local error for (2.1.11), which we shall call Option III, is $O(h^2k + h^3)$. Moreover, since all coefficients in (2.1.11) are positive, the formula is stable for all r.

In some cases it is convenient to replace f_0 by f_4 in (2.1.11), obtaining

$$u_0 \backsim \frac{u_1 + hHu_g + \frac{1}{2r}u_4 + \frac{1}{2}h^2f_4}{1 + hH + \frac{1}{2r}} , \tag{2.1.13}$$

where the numerator in (2.1.12) is changed by a term whose absolute value is $\frac{1}{2}h^2kf_t^*$, where $f_t^* = f_t + f_uu_t$. Since this term is $O(h^2k)$, the order of the local accuracy of (2.1.13) is the same as that of (2.1.11). We will refer to (2.1.13) as Option III'.



Typical Boundary Configuration - Right hand Boundary

Figure 3

We now consider the treatment of (2.1.2b). For the simplest representation we replace u_x by $(u_0 - u_3)h^{-1}$ obtaining

(2.1.14) $\quad u_0 \sim u_3$.

It is easily seen that the leading term of the difference between the right and left members of (2.1.14) is

$$-\frac{h^2}{2} u_{xx}$$

so that (2.1.14) has the same accuracy as Option I. Actually, of course, (2.1.14) is a special case of (2.1.5) where $H = 0$.

Formula (2.1.14) represents the only method which we used for treating (2.1.2b). Perhaps a better procedure would have been to consider (x_0, y_0) as an interior point and to use

(2.1.15) $\quad u_1 \sim u_3$.

Here, the leading term of the error is

$$\frac{h^3}{3} u_{xxx} ,$$

which is $O(h^3)$.

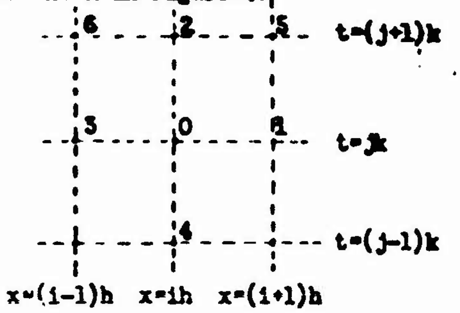
If one does not wish to use the point (x_1,t_1), which is outside the region, one may obtain a formula of the same order of accuracy by using the point (x_4,t_4) instead. The formula may be obtained immediately from (2.1.11) by letting $N = 0$. We get

$$u_0 \sim \frac{u_1 + \frac{1}{2r} u_4 + \frac{1}{2}h^2 f_0}{1 + \frac{1}{2r}} \quad . \tag{2.1.16}$$

The leading term of the error is $O(h^3)$. As before the accuracy would be preserved if we replaced f_0 by f_4 in (2.1.16), obtaining

$$u_0 \sim \frac{u_1 + \frac{1}{2r} u_4 + \frac{1}{2}h^2 f_4}{1 + \frac{1}{2r}} \quad . \tag{2.1.17}$$

2.2 Forward difference method. For the derivation of this and other difference equations it will be convenient to consider a typical configuration of points such as shown in Figure 4.



Typical Configuration - Interior Points

Figure 4

We assume here that $1 \leq i \leq M-1$, but that j may equal zero in which case the point (x_4, t_4) would lie outside of the region.

Probably the simplest difference equation is obtained by replacing u_t by $(u_2-u_0)\,k^{-1}$ and by replacing u_{xx} by $(u_1+u_3-2u_0)\,h^{-2}$. These are familiar finite difference representations of the respective derivatives. However, for this method and for the others given below we shall present a derivation based on methods of numerical quadrature. Although the actual finite difference formulas come out the same, nevertheless it seems simpler to obtain the error estimates by the method which we use.

By (1.1) and the rectangle rule for integration we have

(2.2.1) $$u_2 - u_0 = \int_{t_0}^{t_0+k} u_t\,dt = \int_{t_0}^{t_0+k} (u_{xx} + f)dt = k(u_{xx}+f) + \frac{k^2}{2}(u_{xxt}+f_t^*),$$

where $f_t^* = f_t + f_u u_t$. Using Taylor's series we also obtain

(2.2.2) $$u_{xx} = (u_1 + u_3 - 2u_0)h^{-2} - \frac{1}{12} h^2 u_{xxxx} + \dots \quad .$$

Substituting in the previous equation we get

(2.2.3) $$u_2 \backsim (1-2r)u_0 + r(u_1 + u_3) + kf_0$$

where the leading term of the difference between the right and left members of (2.2.3) is

(2.2.4) $$k\left\{\left(\frac{k}{2} - \frac{h^2}{12}\right)u_{xxxx}\right\}.$$

Since the number of time steps needed to reach a given value of t is proportional to k^{-1} it seems reasonable to take

(2.2.5) $$\left(\frac{k}{2} - \frac{h^2}{12}\right)u_{xxxx}$$

as a measure of the local error. It is well known, (See for instance [18]) that k must satisfy the condition

(2.2.6) $$k \leq \frac{1}{2} h^2$$

in order for the numerical procedure to be stable. Thus the local error for the Forward Difference method is $O(h^2)$.

Of course, such an analysis does not, by any means, indicate that the accuracy in the large is of second order. The question of the order of convergence in the large has been studied for a special class of problems by Juncosa and Young [12]. For an analysis of the situation in the large for the case where $u(x,t)$ is assumed to possess certain partial derivatives in R+S, see Milne [17].

For B V problems with f, ϕ, ψ vanishing, the convergence of the sequence of solutions of the Forward Difference method to the exact solution of the differential equation has been proved by Leutert [16] for $r \leq 1/4$, under the assumption that $g(x)$ is piecewise continuous, that is, continuous except for a finite number of finite jumps, and that $g(x)$ has a one sided first derivative everywhere. Hildebrand [9] proved convergence for $0 < r \leq \frac{1}{2}$ under the assumption that $g(x)$

is piecewise continuous and has bounded variation. More severe restrictions were imposed on $g(x)$ for the case $r = \frac{1}{2}$. Juncosa and Young [11] proved convergence for $r \leq \frac{1}{2}$ under the assumption that $g(x)$ is piecewise continuous.

The numerical computational procedure is extremely simple. Since one knows $u_{i,0}$, $0 \leq i \leq M$, one can compute $u_{i,1}$, $1 \leq i \leq M-1$ using (2.2.3) which for a general point becomes

(2.2.7) $$u_{i,j+1} = (1-2r)u_{i,j} + r(u_{i+1,j} + u_{i-1,j}) + kf_i \,.$$

For B V problems $u_{0,1}$ and $u_{M,1}$ are given and we can proceed at once to the next row. For N D problems we can compute $u_{0,1}$ directly from either (2.1.5), (2.1.7) or (2.1.13) and we can use (2.1.14), (2.1.15) or (2.1.17) to find $u_{M,1}$. Having thus determined all values of $u_{i,1}$ we proceed to compute values of $u_{i,2}$ then $u_{i,3}$ etc.

Were it not for the condition (2.2.6) on the size of k, the Forward Difference method would be ideal for use on computing machines. Unfortunately, when one attempts to improve the accuracy by reducing h, say by a factor of 2, then the number of time steps is increased by a factor of 4, making the total work increase by a factor of 8. More generally, one can assert that the work increases with the third power of h^{-1}, or equivalently, as the third power of M.

2.3 Crank-Nicolson method. If, instead of using the rectangle rule of integration as in the case of the forward difference method, we use the trapezoidal rule, we obtain

(2.3.1) $$u_2 - u_0 = \int_{t_0}^{t_0+k} u_t\,dt = \int_{t_0}^{t_0+k} (u_{xx}+f)\,dt$$

$$= \frac{k}{2}\left[(u_{xx})_0 + (u_{xx})_2 + f_0 + f_2\right] - \frac{k^3}{12}(u_{xxtt} + f^*_{tt}) + \cdots .$$

Using Taylor's series we have

$$(u_{xx})_0 = h^{-2}(u_1 + u_3 - 2u_0) - \frac{h^2}{12}u_{xxxx} + \cdots$$

$$(u_{xx})_2 = h^{-2}(u_5 + u_6 - 2u_2) - \frac{h^2}{12}u_{xxxx} + \cdots .$$

Substituting in (2.3.1) we get

$$u_2 = u_0 + \frac{k}{2h^2}\left[u_1 + u_3 + u_5 + u_6 - 2u_0 - 2u_2\right] + \frac{k}{2}\left[f_0 + f_2\right] - \frac{kh^2}{12}u_{xxxx} - \frac{k^3}{12}(u_{xxtt} + f^*_{tt}).$$

Solving for u_2 we have

(2.3.2) $$u_2 = \frac{1-r}{1+r}u_0 + \frac{r}{2(1+r)}\left[u_1 + u_3 + u_5 + u_6\right] + \frac{k}{2(1+r)}\left[f_0 + f_2\right]$$

where we have neglected the remainder whose leading terms are

(2.3.3) $$\frac{k}{1+r}\left\{-\frac{h^2}{12}u_{xxxx} - \frac{k^2}{12}(u_{xxtt} + f^*_{tt})\right\}.$$

We note that the expression in the braces is $O(h^2)$ as long as $k = O(h)$. Juncosa and Young [13] showed that under certain conditions the error in the large is $O(h^2)$ provided $k = O(h/|\log h|)$. It is not known, however, whether this result can be extended to include cases where $k = O(h)$.

It is sometimes convenient to replace f_2 by f_0 in (2.3.2). The additional error which is introduced is proportional to k^2 and hence the expression in braces occurring in (2.3.3) is increased by a term proportional to k. Thus if k tends to zero like h, the expression in the braces would be $O(h)$ instead of $O(h^2)$.

Another modification of (2.3.2) is obtained by replacing the term $\frac{1}{2}\left[f_0 + f_2\right]$ by $f(x_0, \frac{1}{2}(t_0+t_2), \frac{1}{2}(u_0+u_2))$. This change was made in the program prepared for the Ordvac. For simplicity, we assume here that $f(x,t,u)$ is a function of

u alone. Expanding in a Taylor series about the point $(x,\bar{t})$, where $\bar{t}=\frac{1}{2}(t_0+t_2)$

we have
$$u_2 = \bar{u} + \frac{1}{2}ku_t + \frac{1}{8}k^2u_{tt} + \ldots$$
$$u_0 = \bar{u} - \frac{1}{2}ku_t + \frac{1}{8}k^2u_{tt} + \ldots$$
$$f_2 = \bar{f} + \frac{1}{2}kf_t^* + \frac{1}{8}k^2f_{tt}^* + \ldots$$
$$f_0 = \bar{f} - \frac{1}{2}kf_t^* + \frac{1}{8}k^2f_{tt}^* + \ldots$$

where $\bar{f} = f(\bar{u}), \bar{u} = u(x,\bar{t})$. It is evident that $\frac{1}{2}(f_0+f_2) = \bar{f} + \frac{1}{8}k^2f_{tt}^* + \ldots$

and $\frac{1}{2}(u_0+u_2) = \bar{u} + \frac{1}{8}k^2u_{tt} + \ldots$.

Therefore $f(\frac{1}{2}(u_0+u_2)) = \bar{f} + \frac{1}{8}k^2u_{tt}f_t^* + \ldots$

and $\frac{1}{2}(f_0+f_2) - f(\frac{1}{2}(u_0+u_2)) = \frac{1}{8}k^2f_{tt}^* - \frac{1}{8}k^2u_{tt}f_t^* + \ldots = O(k^2).$

By (2.3.2) and (2.3.3) it is evident that the order of accuracy has not been changed.

For a general point (2.3.2) may be written

$$(2.3.4) \qquad u_{i,j+1} = \frac{1-r}{1+r}u_{i,j} + \frac{r}{2(1+r)}\left[u_{i+1,j} + u_{i-1,j} + u_{i+1,j+1} + u_{i-1,j+1}\right] + \frac{k}{2(1+r)}\left[f_{i,j} + f_{i,j+1}\right].$$

Two alternative numerical procedures are available for using the Crank-Nicolson method. The first is an iterative method wherein, given all values of $u_{i,j}$, $0 \le i \le M$, one chooses a first approximation $u_{i,j+1}^{(0)}$, $0 \le i \le M$ and proceeds to iterate using the formula

$$(2.3.5) \qquad u_{i,j+1}^{(n)} = \omega\left\{\frac{1-r}{1+r}u_{i,j} + \frac{r}{2(1+r)}\left[u_{i+1,j} + u_{i-1,j} + u_{i+1,j+1}^{(n-1)} + u_{i-1,j+1}^{(n)}\right] + \frac{k}{2(1+r)}\left[f_{i,j} + f_{i,j}^{(n)}\right]\right\} - (\omega-1)u_{i,j}^{(n)}, \quad i=1,2,\ldots,M-1.$$

Here the parameter ω known as the <u>relaxation factor,</u> is chosen in the range $1 \le \omega < 2$ to accelerate the convergence. At the end of each iteration one obtains $u_{0,j+1}^{(n)}$ by solving (2.1.5),(2.1.7), (2.1.11), or (2.1.13) for $u_{0,j+1}$ using the latest values of $u_{1,j+1}$ when appropriate. Similarly $u_{M,j+1}^{(n)}$ is obtained. The iterative process is repeated until

$$(2.3.6) \qquad \max_{0 \le i \le M}\left|u_{i,j+1}^{(n)} - u_{i,j+1}^{(n-1)}\right| < \varepsilon$$

where ε is a preassigned tolerance. When the iterative process has converged, one proceeds to determine the values in the next row in the same way.

The choice of the relaxation factor ω_b, which will yield the fastest convergence can easily be computed for linear BV problems. It is shown in [22] that

$$(2.3.7) \qquad \omega_b = 1 + \left[\frac{\mu}{1+\sqrt{1-\mu^2}}\right]^2$$

where μ is the maximum eigenvalue of the $(M-1)x(M-1)$ matrix

$$\begin{pmatrix} 0 & \beta & 0 & 0 & \cdots & 0 \\ \beta & 0 & \beta & 0 & \cdots & 0 \\ 0 & \beta & 0 & \beta & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdot\cdot\beta & 0 \end{pmatrix}$$

where $\beta = \frac{r}{2(1+r)}$ It is easy to show that

$$(2.3.8) \qquad \mu = \frac{r}{1+r}\cos\frac{\pi}{M}\,.$$

Let us now assume that $k = O(h)$ and that, as a result, r is proportional to M. Since for large M we have $1-\mu \sim \frac{1}{M}$, we can show, as in [22], section 4, that the rate of convergence of (2.3.5) varies as $M^{-\frac{1}{2}}$; hence the number of iterations increases as $M^{\frac{1}{2}}$. Since the number of points on each row increases with M, it follows that the work per row is proportional to $M^{3/2}$. Moreover, since the number of time steps increases as M, (instead of M^2 as with the Forward Difference method) we conclude that the total time increases as $M^{5/2}$ with the Crank-Nicolson method compared with M^3 for the Forward Difference method. Thus the Crank-Nicolson method would appear to be superior to the Forward Difference method, at least for large M.

The advantage of the Crank-Nicolson method can be increased still further by using a non-iterative procedure. The effectiveness of this procedure depends on the fact that for a linear system involving a Jacobi matrix, i.e., a matrix whose only non-zero elements occupy the main diagonal and the two diagonals adjacent to the main diagonal, the solution can be obtained very easily. This fact was observed by L.H. Thomas in unpublished notes [21] and was applied to parabolic equations by Bruce, Peaceman, Rachford and Rice [2] . Thus following their discussion, let us suppose we have the system

$$(2.3.9) \quad \begin{cases} B_1 x_1 + C_1 x_2 = d_1 & \\ A_i x_{i-1} + B_i x_i + C_i x_{i+1} = d_i & 2 \leq i \leq M-1 \\ A_{M-1} x_{M-1} + B_M x_M = d_M & \end{cases}$$

If one carries out the familiar Gauss elimination procedure, eliminating the elements below the main diagonal of the matrix, and then uses the method of back substitution, one obtains

$$(2.3.10a) \quad \begin{cases} x_M = q_M & \\ x_i = q_i - b_i x_{i+1} & 1 \leq i \leq M-1 \end{cases}$$

where

$$(2.3.10b) \quad \begin{cases} q_1 = d_1/B_1 \\ q_i = \dfrac{d_i - A_i q_{i-1}}{B_i - A_i b_{i-1}} \end{cases}$$

and

$$(2.3.10c) \quad \begin{cases} b_0 = 0 & \\ b_i = \dfrac{C_i}{B_i - A_i b_{i-1}} & 1 \leq i \leq M-1. \end{cases}$$

The stability of the computation procedure has been investigated by Thomas in [2].

No more than about 3 or 4 times as much work per row would appear to be required with this method as with the Forward Difference method. Thus, if we assume $k = O(h)$, then the amount of work required increases as M^2 instead of $M^{5/2}$ with the iterative procedure and M^3 with the Forward Difference method.

Unfortunately, when (1.1) is non-linear it is not always easy to obtain linear equations and still retain the desired accuracy. For, if we use (2.3.2), we find that the unknown value of $u_{i,j+1}$ is involved, in general, in a non-linear manner in the evaluation of the function $f(x,t,u)$. On the

other hand, if one replaces $u_{i,j+1}$ by $u_{i,j}$, one obtains a less accurate formula, as discussed above. To get around this difficulty in an analogous situation, Bruce, Peaceman, Rachford, and Rice [2] used an iterative procedure together with the non-iterative method for solving the linear equations. Although this procedure might appear to defeat the original purpose of avoiding all iterations, it is reported that convergence is obtained after a very few iterations.

It might not be too unreasonable to preassign a fixed number of iterations, independent of h and k, to be carried out at each time step. Thus, the Heun method for solving ordinary differential equations is the same as the modified Euler method (see for instance, Milne [17]) except that one performs only one iteration. The local error for the Heun method is somewhat larger than, but of the same order of magnitude as the modified Euler method.

Another possible solution to this difficulty would be to use a difference equation involving more points. However, care must be taken to avoid formulas which might lead to unstable procedures, and also to avoid obtaining a matrix which was not a Jacobi matrix.

2.4 <u>DuFort-Frankel method</u>. In [5] DuFort and Frankel obtained a difference equation which is stable for all values of r and which, at the same time, yields explicit formulas for values of u in the new row in terms of values on previous rows. We may derive the formula by use of numerical quadrature as follows.

(2.4.1)
$$u_2 - u_4 = \int_{t_4}^{t_2} u_t\,dt = \int_{t_4}^{t_2} (u_{xx} + f)\,dt$$

$$= 2k\left[u_{xx} + f_0\right] + \tfrac{1}{3}k^3\left[u_{xxt} + f_t''\right] + \dots .$$

Using Taylor's series, as before, we get

$$(2.4.2) \qquad u_2 = u_4 + 2k\left[(u_1 + u_3 - 2u_0)h^{-2} + f_0\right] + \left\{-2k\frac{h^2}{12}u_{xxxx} + \frac{k^3}{3}(u_{xxt} + f_t^e)\right\} + \cdots$$

If we were to neglect the expression in the braces and all higher order terms in (2.4.2) we would obtain Richardson's method [20], which is unstable for <u>all</u> values of r and is thus unsuited for numerical calculations unless very special precautions are taken [15]. Nevertheless, at this point we remark that the local error for Richardson's method is

$$(2.4.3) \qquad k\left\{-\frac{h^2}{6}u_{xxxx} + \frac{k^2}{3}(u_{xxt} + f_t^e)\right\}.$$

In order to obtain a stable method, DuFort and Frankel replace u_0 by $\frac{1}{2}(u_4 + u_2)$ in (2.4.2) and obtain, after solving for u_2,

$$(2.4.4) \qquad u_2 = \frac{1-2r}{1+2r}u_4 + \frac{2r}{1+2r}(u_1 + u_3) + \frac{2k}{1+2r}f_0 + \frac{k}{1+2r}\left\{-\frac{h^2}{6}u_{xxxx} + \frac{k^2}{3}(u_{xxt} + f_t^e) - \frac{2k^2}{h^2}u_{tt}\right\} + \cdots.$$

Now if $k = O(h^2)$, it is evident that the expression in the braces is $O(h^2)$. On the other hand, if $k = O(h)$, then the expression inside the braces need not even tend to zero with h. Consequently, although stability is assured for all h and k, for convergence the ratio k/h must tend to zero with h, (see [5]).

Neglecting remainder terms we have for a general point

$$(2.4.5) \qquad u_{i,j+1} = \frac{1-2r}{1+2r}u_{i,j-1} + \frac{2r}{1+2r}(u_{i+1,j} + u_{i-1,j}) + \frac{2k}{1+2r}f_{i,j}.$$

Evidently, to compute $u_{i,j+1}$, one needs the values of u on the j-th and (j-1)st rows. At the beginning one has only the values for $j = 0$. To obtain the values

of u for $j = 1$ another procedure must be used. This point is discussed further in Section 4.

The convergence of the DuFort-Frankel method for a certain class of problems has been given by Juncosa [14].

2.5. Extrapolation to zero grid size. For the process known as extrapolation to zero grid size, proposed by L. F. Richardson [20], one assumes that the error $e_M(x,t) = u_M(x,t) - u(x,t)$ is proportional to h^{α}. Here $u(x,t)$ is the exact solution of the differential equation and $u_M(x,t)$ is the solution of the difference equation with $h = A/M$. Usually certain restrictions are assumed for k. Even though $e_M(x,t)$ might vanish like $M^{-\alpha}$, as is indeed true for certain cases as shown in [12], [13], nevertheless it does not necessarily follow that $e_M(x,t)$ is exactly proportional to $M^{-\alpha}$. This can be easily seen by considering the case where $e_M(x,t) = M^{-\alpha} \sin M$.

If $e_M(x,t)$ were directly proportional to $M^{-\alpha}$, one could determine $u(x,t)$ in terms of $u_M(x,t)$ and $u_{2M}(x,t)$. Indeed we would have

$$u(x,t) = u_{2M}(x,t) + \frac{1}{2^{\alpha}-1}\left[u_{2M}(x,y) - u_M(x,y)\right]. \tag{2.5.1}$$

If one solves the difference equation for three different mesh sizes, obtaining $u_M(x,y)$, $u_{2M}(x,y)$, and $u_{4M}(x,y)$ we can perform a simple test for the assumption that $e_M(x,y)$ is proportional to $M^{-\alpha}$. Indeed, if the assumption were valid, we would have

$$\frac{e_M(x,y)}{e_{2M}(x,y)} = 2^{\alpha}. \tag{2.5.2}$$

By computing the ratio, ρ, appearing in (2.5.2) one can determine the proper extrapolation to zero grid size, provided such exists, and also estimate the

degree of convergence of the sequence of solutions of the difference equation. It is not necessary to compute a itself since 2^a appears in (2.5.1). We wish to emphasize once more, that the considerations of this section are not rigorous.

3. Specific Problems Considered

In order to obtain some indication as to the validity of the non-rigorous mathematical considerations of the previous section, we have solved some sample problems on the Ordvac computer at Aberdeen. We chose problems involving either boundary conditions (2.1.1) or (2.1.2) and either linear and non-linear differential equations. Actually, as will be seen in the next section, the program which was prepared for Ordvac is capable of handling entire classes of problems rather than merely the specific problems described below.

3.1. Boundary value problem - linear differential equation (BVL). Here we selected a problem for which an analytic solution is readily available. The equations are

$$(3.1.1) \qquad \begin{cases} u_t = u_{xx} & , \ 0 < x < 1, \quad t > 0, \\ u(x,0) = g(x) & , \ 0 < x < 1 \\ u(0,t) = u(1,t) = 0 & , \ t \geq 0. \end{cases}$$

The analytic solution for the case $g(x) \equiv 1$ is given by

$$(3.1.2) \qquad u(x,t) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin n\pi x \, e^{-n^2\pi^2 t}$$

We shall, by solving the difference equations with various mesh sizes, attempt to determine the rate of convergence for both methods. For the Crank-Nicolson method we shall let $k = O(h)$ as $h \to 0$, even though the theoretical results have been proved only under stronger restrictions on k, [13], in an attempt to determine whether a generalization is to be expected.

3.2. Boundary value problem - non-linear differential equation (BVN). The following problem described by Hartree [7] appears to be typical of this class of problems.

$$(3.2.1)\qquad \begin{cases} u_t = u_{xx} + \gamma e^u & 0 < x < 1, \quad t > 0 \\ u(x,0) = 0 & 0 < x < 1 \\ u(0,t) = u(1,t) = 0 & t \geq 0 \end{cases}$$

Here $u(x,t)$ represents the temperature in a substance where there is generation of heat increasing exponentially with the temperature. When γ exceeds a certain critical value there is no steady state solution and the solution increases beyond all bounds. We shall not study this aspect of the problem but, rather, will use a fixed value of 0.8 for γ, which is below the critical value. We shall solve the problem by the Forward Difference and Crank-Nicolson method for several values of h. Solutions for various values of γ have been obtained by Hartree using a differential analyzer and the results agreed quite well with experiments.

3.3. Normal derivative problems (NDL and NDN). Here we consider boundary conditions (2.1.2) with both the linear equation

$$(3.3.1)\qquad u_t = u_{xx} \qquad 0 < x < A, \quad t > 0$$

and the non-linear equation

$$(3.3.2)\qquad u_t = u_{xx} + e^{-1/u}, \qquad 0 < x < A, \quad t > 0.$$

The initial conditions are

$$(3.3.3)\qquad f(x) = c \qquad 0 \leq x \leq A$$

where in the cases which we shall consider $c = .031$. Actually, in our computations, we shall take for our initial function not $g(x) \equiv .031$ but a set of

values obtained for a later value of t using the analytic solution of the linear problem (see for instance [8], page 22). The values are given in Table I. Even in the non-linear case, these values are sufficiently accurate since the non linear term is negligible for small values of t. This adjustment in the initial conditions was made to eliminate any possible effect of the singularity caused by the discontinuity of u_x when $t = 0$. It is by no means certain that such a precaution would be necessary, and for the linear case at least, it should be possible to determine analytically the effect of the singularity.

Problems of this type involving (3.3.1) are designated MDL and those involving (3.3.2) are designated MDM. For MDM problems the solution $u(x,t)$ represents the temperature in a propellant where the reaction rate is determined by the Arrhenius term $e^{-1/u}$. The propellant is assumed to be a semi-infinite slab, $0 \leq x < \infty$. The propellant is heated by a gas at temperature u_g which occupies the region $x < 0$ and whose influence on the propellant depends on the heat transfer coefficient h. In order to be able to use finite difference methods we consider a slab of finite thickness A and replace the condition $\lim_{x \to \infty} u_x = 0$ by the condition (2.1.2b). Further details on the physical motivation for these problems may be found in [8].

In order to represent more accurately the physical situation, it was sometimes necessary to introduce "shutoff", that is after a certain value of t, u_g is greatly reduced. However, we shall not do this in our study.

We shall also not consider the determination of the "ignition time" which is the time required for the value of $u(x,t)$ to increase above a preassigned value.

We shall assume the following values for the parameters

$A = 73{,}600$

$c = .031$

$u_g = .18$

$H = 2\times10^{-8}$

Prior to 1952 a number of calculations were done on this problem at the Aberdeen Proving Ground using the Bell Relay Computer and the Eniac. The Forward Difference method was used with $r = \frac{1}{2}$ and several values of h. We shall make use of some of these results in Section 5.

4. Description of the Program for Univac

To run a particular problem with the program as presently coded, it is necessary to specify three major conditions; i.e. (1) the difference equation to be used, (2) the boundary conditions, and (3) the necessary parameters such as h, r, etc.

All of the difference equations and boundary conditions described in Section 2 have been coded and any problem involving these can be computed using the program. Except for the CNI method, the interior points of the (j+1)st row are computed followed by the computation or insertion of the boundary values of that row. This permits the boundary conditions to be easily modified. The appendix contains the detailed equations used in the present coding.

Unless otherwise instructed, the machine will assume that the differential equation is linear. In the non-linear case it is necessary to instruct the machine to compute the non-linear term.

For the DF method, since the j-th and (j-1)st rows are necessary to compute the (j+1)st row (see appendix), it was necessary to compute the first row by a different method. We used the FD method. The number of rows necessary to be computed by the FD method depends upon r, which involves k, the increment in time. This is noted in the Appendix.

For the CN method, NL case, when the values of the function became large, the iterative scheme would not converge. A feature was added in the program whereby if the number of iterations exceeded a given K_0 for the (j+1)st row, the coding would revert back to the FD method and commence computation from the j-th row using r', where $r = r' \cdot 2^{s}$ and $0 < r' \leq \frac{1}{2}$.

Shutoff, another feature, can be described as reducing the gas temperature

(u_g) or shutting off the gas flow after time t_s (see Section 3). After computing j_s rows (ND only), the value of u_g can be changed for the (j_s+1)st row and the remaining rows.

The parameters that must be chosen are:

$h = \Delta x/A$

$r = k/h^2$

S switch - which is the stopping procedure where $S = S_1$ stops the machine after m rows and $S = S_2$ stops machine when max $|u_{i,j}|$ exceeds or falls below a given tolerance.

Besides these parameters, certain specific cases will require a specification of other information, such as:

X_0(CN) - as explained above

ω (CN) - the relaxation factor. If ω is not specified, $\omega = 1$ is assumed

γ (BVN): (.8 was used)

H (ND) : ($2 \cdot 10^{-6}$ was used)

u_g (ND): (.18 was used)

A (ND) : (73,600 was used)

j_s, $u_g^{(1)}$ (ND) - (shutoff); replaces u_g by $u_g^{(1)}$ after computing j_s rows.

The non-linear term was computed by a floating point subroutine with the exponential factor being computed by taking the first 15 terms of the series expansion of e^x. Initially, the exponent was normalized; i.e. placed in the form $a \cdot 2^a$ where $\frac{1}{2} \leq |a| < 1$, or $a = 0$ where appropriate. (For the BV cases, $a = 4$ since all values were scaled by 2^{-4}. For ND cases, the presence of the exponent $(-\frac{1}{u})$ necessitated

the normalizing procedure.) Then, $2^{-\alpha} e^{a}$ was computed followed by a normalization of e^{a}. Since $e^{f(u)}$ was desired, the scaling problem was solved by the following:

$$e^{f(u)} = e^{a \cdot 2^{\alpha}} = (e^{a})^{2^{\alpha}}$$

Therefore, it was necessary to square e^{a} α times in computing the non-linear term.

5. Analysis of Results

In Tables II - IX we give the values of $u(x,t)$ for the respective problems as obtained by the various finite difference methods. Scaling factors are indicated on the tables. In each case the columns headed FD($\frac{1}{2}$) and FD($\frac{1}{4}$) contain results obtained using the Forward Difference method with $r = \frac{1}{2}$ and $r = 1/4$, respectively. Columns headed CN(I) and CNN(I) contain the results obtained by using the Crank-Nicolson method using the iterative and non-iterative numerical procedures, respectively. For different values of $M = A/h$, the following values of r are used

M	r
8	1/2
16	1
32	2
64	4

The column headed CNN(II) contains results obtained using the Crank-Nicolson method, non-iterative and with the following values of r

A/h	r
8	1
16	2
32	4
64	8

The value of t can be obtained from the row number by multiplying the row number by $\frac{1}{128}A^2$. Thus the row number equals t/k_8 where k_8 is the value of k corresponding to $M = 8$ and $r = 1/2$. The column headed "Eniac" gives results obtained on the Eniac using the Forward Difference method and $r = 1/2$.

Corresponding to each of the row numbers 4, 8, 12, 16, and 20 and to the indicated values of x/A we give the following results, when available,

M	
8	u_8
	e_{16}
16	u_{16}
	A/B
32	u_{32}
	C/D
64	u_{64}
	[E]

Here u_M represents the solution of the difference equation being considered with $h = A/M$. Also, $e_{16} = u_{16} - u_8$, $A = e_{16}/e_{32}$, $B = e_{32} = u_{32} - u_{16}$, $C = e_{32}/e_{64}$, and $D = e_{64} = u_{64} - u_{32}$. In the brackets we give the extrapolated value. The estimated value of ρ used in the extrapolation is given at the bottom of the column, (see section 2.5). For the extrapolation e_{64} is used when available, otherwise e_{32} is used. An asterisk in place of $e_{M/2}/e_M$ indicates that the computed value is not regarded as significant, usually because e_M is too small. An asterisk near an extrapolated value means that the value obtained from the use of the smallest mesh size is given in place of the extrapolated value.

5.1. Boundary value linear (BVL). In this case the results obtained from the Crank-Nicolson method with the iterative and the non-iterative numerical procedures were identical. Since the ratios e_M/e_{2M} are close to 4 it would appear that $e_M = O(M^{-2})$ and that extrapolation to zero grid size based on $\alpha = 2$ might give good results. Indeed, if one uses this procedure one obtains almost identical results for all methods and very close agreement with the exact value as calculated from (3.3.2). We note that the values obtained from the Forward Difference method with $r = \frac{1}{2}$ and $M = 32$ are considerably different

from the exact values and that the extrapolation process has greatly improved the accuracy. We also remark that for this particular problem the Crank-Nicolson method is much more accurate than the Forward Difference method although the extrapolated values are practically the same.

5.2. Boundary value non-linear (BVN). As in the linear case the values of e_M/e_{2M} are, with the FD and CN methods, close to 4. Extrapolation to zero grid size using $\rho = 4$ and $\alpha = 2$ yields results which are nearly the same for both the FD method (with $r = \frac{1}{2}$ as well as with $r = 1/4$) and for CN(I). On the other hand with the non-iterative procedures CNM(I) and CNM(II), (where the less accurate difference equation obtained by replacing f_2 by f_0 in (2.3.2) was used), the values of e_M/e_{2M} were usually between 3 and 3.5. Thus as expected, the order of convergence was less than for the other methods. Nevertheless with an extrapolation using $\rho = 3$ we obtained fairly accurate results, especially for CNM(I).

We believe that combining CNM with a few iterations, not more than five, would increase the order of convergence.

The error due to using f_0 instead of f_2 in (2.3.2) varies approximately as M^{-1} as can be seen from comparing results of CN(I) and CNM(I). Thus for the 10th row and for $x = \frac{1}{2}$ we have

M	CN(I)	CNM(I)	CN(I)-CNM(I)
8	72534	72400	134
16	72888	72821	67
32	72977	72943	34

(All values have been multiplied by 10^5.)

Examining the differences 134, 67, 34 we observe that they are approximately in the ratios 4:2:1. Hence the differences appear to tend to zero like M^{-1}. This tends to confirm our prediction that the error introduced by using the less accurate difference equation is of order M^{-1}.

5.3. Normal derivative linear (NDL). In both the linear and the non-linear cases we are concerned with three options for representing the condition (2.1.2a). We first consider the results obtained for the linear case for Options I, II, III as given in Tables IV, V, VI respectively. As in the boundary value linear case it did not matter whether one used the iterative or the non-iterative procedure for the Crank-Nicolson method.

For Option I the values obtained by the Forward Difference method for $r = 1/4$ and $r = 1/2$ agree closely with each other and with those obtained with CNN(I) and CNN(II). Based on the observed value of ρ, extrapolated values were obtained as indicated. Of course, in this case we can conclude little from the fact that the extrapolated values obtained from the different methods agree with each other. However, the extrapolated values do agree rather closely with those obtained from results using Option II and III, and the latter two agree with each other. Moreover, the difference $u_{64} - u_{32}$ obtained for CNN(I) with Option III was very small, and this indicates that u_{64} is probably very close to the true value of u. Thus it would appear that in this case (2.1.5) (Option I) is a reasonably satisfactory representation of (2.1.2a) provided one uses the proper extrapolation procedure.

Finally we compare our results with the "exact" values obtained under the assumption that A is infinite, using the procedure given in [8]. These values are given on Table IV in braces. Near the line x = A our values will not agree

with these "exact" values, because in our case A is finite. Nevertheless the agreement is generally rather close.

We give in Table V the values obtained on the Eniac using Option II and the FD method with $r = \frac{1}{2}$. Although the Eniac values and the Ordvac values listed under FD, again with $r = \frac{1}{2}$, agree exactly for $M = 8$, exact agreement cannot be expected for other values of M. The reason for this is that A=64,400 was used for the Eniac results instead of A=73,600 as used on the Ordvac. On the Eniac, instead of (2.1.14), the condition

$$u_M = u_{M+1}$$

was used to represent (2.1.2b). Thus, since 64,400/73,600 = 7/8 we have exact agreement in the case $M = 8$.

Near the left hand boundary, ρ appears to be rather close to 4.0 so that we use extrapolation based on $\alpha = 2$. We note that $u_{32} - u_{16}$ is small, and this indicates that u_{32} is already quite accurate. For $x/A = \frac{1}{2}$ the use of $\rho = 3$ seems indicated, and for $x/A = 1$ we used $\rho = 2$. It appears that the largest errors occurred at the right hand boundary.

Similar results were obtained for Option III as indicated in Table V. Of course for Option III it was possible to use CNN(I) and thus use many fewer time steps than were necessary for the Forward Difference method.

It seems reasonable to suppose that, by using either (2.1.15) or (2.1.17) to represent more accurately the right hand boundary condition, and by using Option III or III', the ratio ρ would be nearly 4 everywhere and that extrapolation based on $\alpha = 2$ could have been used for all points.

5.4. Normal derivative non-linear (NDN). The Eniac results obtained

using the FD method for Option II indicate that extrapolation to zero grid size using $\alpha = 2$ for $x/A = 0$, $1/4$ and $\alpha = 1$ for $x/A = 1/2$, 1 should be quite accurate. The same also appears to be true for our Ordvac results. Of course the extrapolated values obtained in the respective cases do not agree because of the difference in the boundary conditions at $x/A = 1$, as mentioned above. The extrapolated values obtained on the Ordvac using Option II and the FD method also agree very closely with those extrapolated values obtained using CN(1) with Option III. It is difficult to make definite statements as to the behavior of the CNN method except that the errors are small and irregular. Here, of course, Option III' was used.

For Option I the ratios ρ are all approximately 2 with all difference equations as in the linear case. The errors are somewhat larger and the extrapolated values somewhat less accurate than with Option III.

5.5. Machine times. Figures 5 and 6 show the approximate times required to compute $t = 10k_8$ rows by the various methods in both the linear and in the non-linear cases respectively. The times for BV problems and for ND problems were nearly the same. The slopes for the curves corresponding to the FD, CN and CNN methods, after proper corrections have been made to account for the use of different logarithmic bases, are approximately 3, 5/2, and 2 respectively. Hence as expected, the times increase as M^3, $M^{5/2}$ and M^2 respectively. For the non-linear cases about 10 times as much machine time was required as for the linear cases.

The number of iterations required using the Crank-Nicolson method seemed to be nearly independent of whether the differential equation was linear or non-linear or whether BV or ND boundary conditions were used. The following

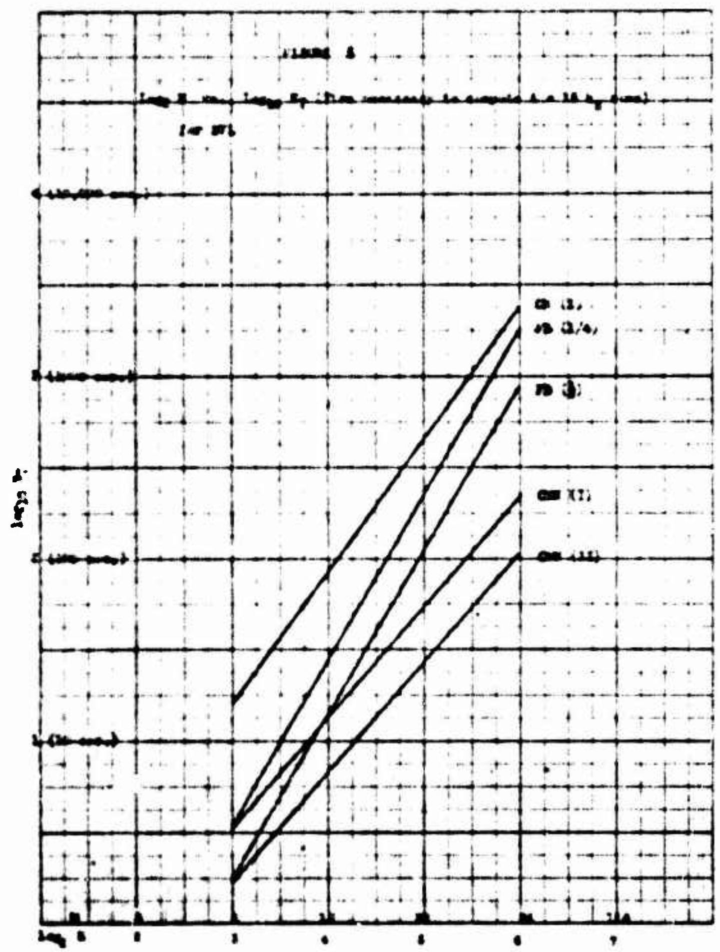


FIGURE 5

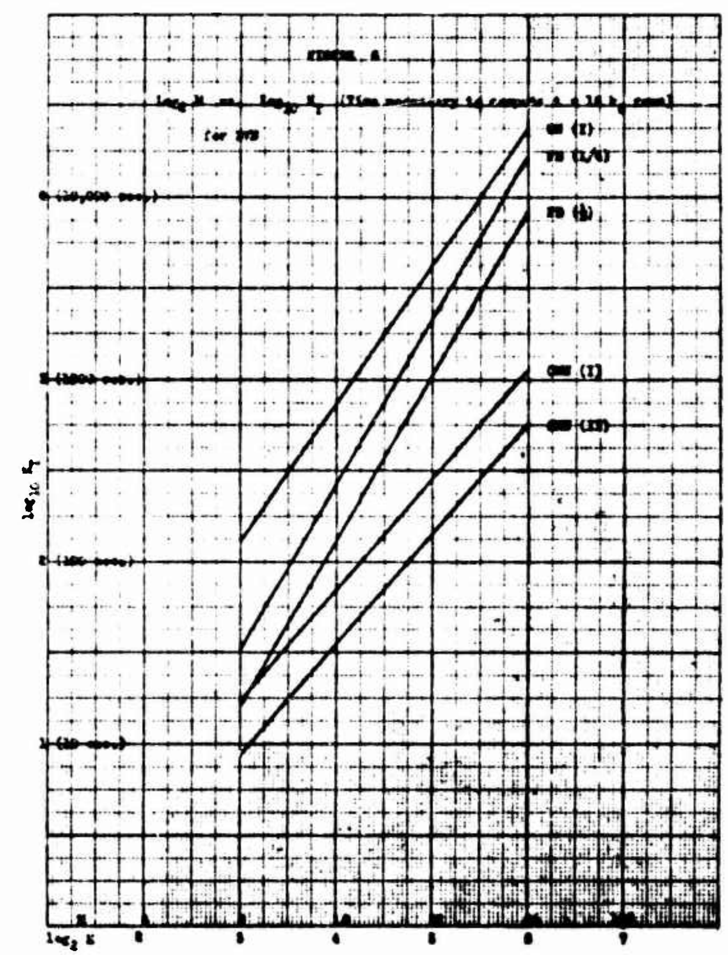


FIGURE 6

table gives the value of ω used for each M and the approximate number of iterations required per row.

M	ω	No. of Iterations Per Row
8	1.025	8
16	1.069	11
32	1.144	15
64	1.375	20

5.6. Tentative conclusions. Considering the analysis of the results as a whole we are led to the following tentative conclusions:

1. In the linear case, the error caused by the use of any of the difference equations rather than the differential equations varies as h^2. In the non-linear case this is true for the FD and CN methods but for the CNN(I) and CNN(II) methods the error varies as h.

2. The use of Option I and the condition (2.1.14) used at $x/A = 1$ introduce errors of order h. On the other hand, the use of Option II, when it can be used, and Options III and III' introduces errors of order h^2.

Further work should include the following:

1. In the non-linear cases, BVN and NDN, try CNN with a fixed number of iterations independent of h.

2. For NDN use CNN with Option III' for the left hand boundary condition and (2.1.15) or (2.1.17) to represent the right hand boundary conditions.

5.7. Dufort-Frankel method. We have not given the results which were obtained using the DFF method. The results which were obtained indicated that the method would not be as effective as the CN or CNN method. However, this should be studied further both analytically and by running some cases on the

machine. Results obtained in [5] indicate that one must require that $k = o(h)$ in order to obtain convergence, and that, moreover, the order of convergence is dependent on the order of h/k as $h \to 0$. Also, the fact that a special procedure appears to be needed to start the process is a disadvantage of the method.

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List of Abbreviations

1. Differential Equations and Boundary Conditions

Symbol	The function $f(x,t,u)$ in (1.1)	Boundary conditions
HVL	0	See (3.1.1) with $g(x) \equiv 1$.
HVN	e^u	See (3.2.1).
NDL	0	See (2.1.2).
NDN	$e^{-1/u}$	See (2.1.2).

For both NDL and NDN the initial values used are given in Table I.

2. Difference Equations

FD Forward Difference

FD($\frac{1}{2}$) uses $r = \frac{1}{2}$

FD($\frac{1}{4}$) uses $r = \frac{1}{4}$

CN Crank-Nicolson (iterative)

CN(I) uses $r = \frac{1}{2}, 1, 2, 4$ with $M = 8, 16, 32, 64$, respectively.

CNN Crank-Nicolson (non-iterative)

CNN(I) uses $r = \frac{1}{2}, 1, 2, 4$ with $M = 8, 16, 32, 64$ respectively.

CNN(II) uses $r = 1, 2, 4, 8$ with $M = 8, 16, 32, 64$ respectively.

DF Dufort-Frankel

3. Representations of (2.1.2a) - (for ND problems only)

Option	Equation
I	(2.1.5)
II	(2.1.7)
III	(2.1.11)
III'	(2.1.13)

TABLE I

Initial values for ND cases
(M = 64)

x/A	$u_{i,0} \cdot 10^7$	x/A	$u_{i,0} \cdot 10^7$	x/A	$u_{i,0} \cdot 10^7$
0	*** 34304		* 31123		** 31000
	33980		31100		31000
	* 33677	3/8	*** 31081		* 31000
	33395		31065		31000
	** 33134		* 31052	3/4	*** 31000
	32894		31041		31000
	* 32674		** 31033		* 31000
	32473		31026		31000
1/8	*** 32290		* 31020		** 31000
	32125		31016		31000
	* 31977	1/2	*** 31012		* 31000
	31844		31009		31000
	** 31726		* 31007	7/8	*** 31000
	31621		31005		31000
	* 31529		** 31004		* 31000
	31449		31003		31000
1/4	*** 31378		* 31002		** 31000
	31318		31002		31000
	* 31265	5/8	*** 31001		* 31000
	31220		31001		31000
	** 31182		* 31001	1	*** 31000
	31150		31001		

* - M = 32

** - M = 32, 16

*** - M = 32, 16, 8

TABLE II

SOLUTIONS OF DIFFERENCE EQUATION FOR BVL
(All values multiplied by 10^5.)

		x = 1/4, 3/4					x = 1/2			
Row	M	FD(¼)	FD(½)	CNN(I)*	CNN(II)*		FD(¼)	FD(½)	CNN(I)	CNN(II)
	8	17520	18823	19376	19333		24777	26619	27402	27341
		1303	328		-54		1842	464		-77
20	16	18823	19151		19279		26619	27083		27264
		3.97/328	4.0/82		3.86/-14		3.97/464	4.0/116		3.95/-20
	32	19151	19233	19268	19265		27083	27199	27248	27244
		[19260]	[19260]		[19260]		[27238]	[27238]		[27237]
	EXACT	[19260]					[27238]			
	8	24048	25674	26278	26230		34009	36084	37159	37094
		1626	409	-44	-8		2299	801	-60	-11
16	16	25674	26083	26234	26222		36308	36885	37099	37083
		3.98/409	4.0/102	4.0/-11	4.0/-2		3.98/577	5.5/145	3.8/-16	2.75/-4
	32	26083	26185	26223	26220		36885	37030	37083	37079
		[26219]	[26219]	[26219]	[26219]		[37077]	[37079]	[37076]	[37079]
	EXACT	[26219]					[37078]			
	8	33008	35023	35647	35588		46680	49520	50377	50316
		2015	506		81		2840	708		113
12	16	35023	35529		35669		49520	50228		50429
		3.98/506	3.98/127		3.68/22		4.01/708	4.0/177		4.35/26
	32	35529	35656	35695	35691		50228	50405	50459	50455
		[35698]	[35698]		[35698]		[50464]	[50464]		[50464]
	EXACT	[35698]					[50464]			
	8	45312	47839	48485	48322		64063	67451	68014	68025
		2527	646		278		3388	822		327
8	16	47839	48485		48600		67451	68273		68452
		4.0/630	3.99/162		3.48/80		4.12/822	4.0/204		4.67/70
	32	48469	48647	48688	48680		68273	68477	68519	68522
		[48679]	[48691]		[48707]		[68547]	[68545]		[68545]
	EXACT	[48701]					[68545]			
	8	62500	66534	67581	66330		87500	90192	89783	90477
		4034	1095		1517		2692	537		370
4	16	66534	67629		67847		90192	90729		90847
		3.68/1095	3.95/277		14.87/102		5.01/537	4.16/129		11.56/32
	32	67629	67906	67982	67943		90729	90858	90820	90879
		[67994]	[67996]		[67977]		[90908]	[90901]		[90890]
	EXACT	[67999]					[90900]			
	c	4	4	4	4		4	4	4	4

* - See Section 5

TABLE III

SOLUTIONS OF DIFFERENCE EQUATION FOR BVN
(All values multiplied by 10^5.)

Rem	N	x=1/4, 3/4: FD(½)	FD(1/4)	CN(I)*	CNN(I)*	CNN(II)*	x=1/2: FD(½)	FD(1/4)	CN(I)	CNN(I)	CNN(II)
	8	62982	62465	61990	61900	61944	82941	82209	81558	81410	81330
		-524	-130	226	272	292	-738	-180	324	389	417
20	16	62458	62335	62216	62172	62135	82203	82029	81862	81799	81747
				3.97/57	3.5/78	3.06/95			4.0/81	3.5/112	3.1/135
	32			62273	62250	62230			81943	81911	81882
				[62292]	[62289]	[62278]			[81970]	[81967]	[81950]
	8	56686	56128	55625	55530	55471	74034	73245	72554	72400	72316
		-557	-130	247	295	316	-783	-180	354	421	451
16	16	56129	55998	55872	55825	55787	73251	73065	72908	72821	72767
		4.25/-131	3.0/-33	4.0/62	3.5/85	3.1/102	4.26/-184	4.1/-44	4.0/89	3.45/122	3.1/146
	32	55998	55965	55934	55910	55889	73067	73021	72977	72943	72913
		[55954]	[55954]	[55955]	[55952]	[55940]	[73006]	[73006]	[73007]	[73004]	[72986]
	8	48288	47718	47218	47125	47068	62156	61351	60646	60514	60432
		-562	-123	252	298	318	-789	-169	360	426	455
12	16	47726	47596	47470	47423	47386	61367	61182	61006	60940	60887
		4.36/-129	4.1/-30	4.0/63	3.43/87	3.1/103	4.35/-181	4.1/-42	4.0/90	3.45/123	3.1/147
	32	47597	47565	47533	47510	47488	61155	61140	61096	61063	61034
		[47554]	[47555]	[47554]	[47554]	[47541]	[61126]	[61126]	[61126]	[61125]	[61109]
	8	37084	36550	36099	36018	35978	46312	45576	44965	44851	44770
		-522	-108	231	271	280	-714	-141	317	374	406
8	16	36562	36442	36330	36289	36258	45598	45435	45282	45225	45176
		4.46/-117	4.15/-26	4.0/58	3.5/78	3.1/91	12.31/-58	4.15/-34	3.0/81	3.4/109	3.1/131
	32	36445	36416	36388	36367	36349	45540	45401	45363	45334	45307
		[36406]	[36407]	[36407]	[36409]	[36398]	[45521]	[45500]	[45390]	[45354]	[45373]
	8	22082	21593	21231	21176	21224	25237	24815	24499	24477	24230
		-478	-91	181	209	151	-405	-66	170	155	204
4	16	21604	21502	21412	21385	21375	24832	24749	24669	24632	24583
		4.P/-100	4.14/-22	3.9/46	8/-10	2.52/60	5.19/-78	4.4/-15	3.8/45	2.46/63	3.1/31
	32	21504	21480	21458	21375	21435	24754	24734	24714	24695	24574
		[21471]	[21475]	[21473]	[21370]	[21465]	[24728]	[24729]	[24729]	[24727]	[24720]
	ρ	4	4	4	3	3	4	4	4	3	3

* = See Section 5

TABLE IV

SOLUTIONS OF DIFFERENCE EQUATIONS FOR [illegible], OPTION I.
(All values multiplied by 10^5.)

$x/a = 0$

Row	N	FD(r=1/4)	FD(r=1/2)	CNM(I)*	CNM(II)*
	8	41833	41837	41815	41789
		-524	-531	-518	-499
20	16	41309	41306	41297	41290
		2.05/-255		2.1/-247	2.06/-242
	32	41054		41050	41048
				2.04/-121	
	64			40929	
	EXACT	[40799] [40806]		[40808]	[40806]
	8	40941	40943	40917	40889
		-519	-524	-509	-488
16	16	40422	40419	40408	40401
		2.06/-252	2.1/-250	2.1/-242	2.06/-237
	32	40170	40169	40166	40164
				2.03/-119	
	64			40047	
	EXACT	[39918] [39928]	[39919]	[39928]	[39927]
	8	39937	39939	39908	39875
		-514	-520	-501	-476
12	16	39423	39419	39407	39399
		2.07/-248	2.11/-246	2.1/-237	2.1/-231
	32	39175	39173	39170	39168
				2.04/-116	
	64			39054	
	EXACT	[38927] [38942]	[38927]	[38938]	[38937]
	8	38755	38759	38724	38683
		-503	-510	-490	-459
8	16	38252	38249	38234	38224
		2.1/-240	2.13/-239	2.2/-228	2.04/-220
	32	38012	38010	38006	38004
				2.07/-110	
	64			37896	
	EXACT	[37772] [37780]	[37771]	[37786]	[37784]
	8	37247	37262	37220	37162
		-467	-484	-460	-416
4	16	36780	36778	36760	36746
		2.12/-220	2.2/-219	2.2/-206	2.1/-196
	32	36560	36559	36554	36550
				2.10/-98	
	64			36456	
	EXACT	[36340] [36361]	[36340]	[36358]	[36354]
	ρ	2	2	2	2

$x/a = 1/4$

Row	N	FD(r=1/4)	FD(r=1/2)	CNM(I)	CNM(II)
	8	37176	37184	37156	37140
		-336	-345	-335	-316
20	16	36840	36839	36831	36824
		2.05/-164		2.12/-158	2.08/-152
	32	36676		36673	36672
				2.00/-76	
	64			36597	
	EXACT	[36512] [36510]		[36521]	[36520]
	8	36294	36302	36280	36252
		-307	-317	-304	-283
16	16	35987	35985	35976	35969
		2.05/-150	2.14/-148	2.1/-142	2.1/-137
	32	35837	35837	35834	35832
				2.06/-59	
	64			35765	
	EXACT	[35687] [35695]	[35689]	[35696]	[35695]
	8	35319	35328	35304	35272
		-270	-281	-266	-242
12	16	35049	35047	35038	35030
		2.06/-131	2.16/-130	2.2/-124	2.1/-118
	32	34918	34917	34914	34912
				2.07/-60	
	64			34854	
	EXACT	[34787] [34795]	[34787]	[34794]	[34794]
	8	34200	34216	34190	34151
		-211	-228	-212	-182
8	16	33989	33988	33978	33969
		2.03/-104	2.2/-103	2.2/-95	2.05/-89
	32	33885	33885	33882	33880
				2.09/-46	
	64			33836	
	EXACT	[33781] [33792]	[33782]	[33790]	[33791]
	8	32844	32883	32862	32806
		-93	-129	-117	-73
4	16	32751	32754	32745	32733
		1.69/-55	2.22/-58	2.29/-51	1.74/-42
	32	32696	32696	32694	32691
				2.13/-24	
	64			32670	
	EXACT	[32641] [32646]	[32638]	[32646]	[32649]
	ρ	2	2	2	2

$x/a = 1/2$

Row	N	FD(r=1/4)	FD(r=1/2)	CNM(I)	CNM(II)
	8	34183	34205	34203	34181
		-194	-213	-214	-197
20	16	33989	33992	33989	33984
		2.04/-95		2.23/-96	2.14/-92
	32	33894		33893	33892
				2.13/-45	
	64			33848	
	EXACT	[33799] [33758]		[33803]	[33800]
	8	33448	33471	33470	33448
		-146	-166	-168	-151
16	16	33302	33305	33302	33297
		1.97/-74	2.16/-77	2.24/-75	2.13/-71
	32	33228	33228	33227	33226
				2.14/-35	
	64			33192	
	EXACT	[33154] [33140]	[33151]	[33157]	[33155]
	8	32708	32735	32736	32713
		-94	-118	-121	-104
12	16	32614	32617	32615	32609
		1.81/-52	2.19/-54	2.28/-53	2.17/-48
	32	32562	32563	32562	32561
				2.12/-25	
	64			32537	
	EXACT	[32510] [32509]	[32502]	[32512]	[32513]
	8	31973	32006	32012	31989
		-36	-54	-71	-53
8	16	31937	31952	31941	31936
		1.49/-26	2.12/-30	2.44/-29	2.04/-26
	32	31911	31922	31912	31910
				2.07/-14	
	64			31898	
	EXACT	[31885] [31882]	[31892]	[31884]	[31884]
	8	31106	31145	31163	31160
		26	-8	-23	-23
4	16	31132	31137	31140	31137
		*/-1	1.33/-5	2.88/-8	2.6/-5
	32	31131	31132	31132	31132
				2.67/-3	
	64			31129	
	EXACT	[31130] [illegible]	[31127]	[31126]	[31127]
	ρ	2	2	2	2

* - see Section 5

TABLE IV

SOLUTIONS OF DIFFERENCE EQUATIONS FOR [illegible], OPTION I.
(All values multiplied by 10[illegible].)

	x/h = ½				x/h = ¾					x/h = 1				
…½)	FD(r-½)	CNN(I)	CNN(II)		FD(r-½)	FD(r+½)	CNN(I)	CNN(II)		FD(r-½)	FD(r-½)	CNN(I)	CNN(II)	
…76 …36 …40 …-168 …76	37184 -345 36839	37156 -335 36831 2.12/-158 36673 2.00/-76 36597 [36521]	37140 -316 36824 2.08/-152 36672 [36520]		34183 -194 33989 2.04/-95 33894 [33799] [33758]	34205 -213 33992	34203 -214 33989 2.73/-96 33893 2.13/-45 33848 [33803]	34181 -197 33984 2.14/-92 33892 [33800]		32360 -229 32131 2.24/-10[illegible] 32030 [illegible]	32387 -259 32128	32417 -275 32142 2.5/-110 32032 2.24/-49 31983 [31934]	32399 -261 32138 2.44/-107 32031 [illegible]	
…4 …07 …17 …150	36302 -317 35985 2.14/-148 35837 [35689]	36280 -304 35976 2.1/-140 35836 2.06/-59 35765 [35696]	36252 -283 35969 2.1/-131 35838 [35697]		33418 -116 33302 1.97/-74 33228 [33154] [33140]	33471 -166 33305 2.16/-77 33228 [33151]	33470 -168 33302 2.24/-75 33227 2.14/-35 33192 [33157]	33448 -151 33297 2.13/-71 33226 [33155]		31822 -146 31676 2.21/-66 31610 [31544] [31278]	31861 -178 31683 2.52/-71 31612 [31541]	31884 -196 31688 2.61/-75 31613 2.87/-33 31580 [31547]	31869 -184 31612 2.52/-73 31612 [31539]	
…31	35328 -281 35047 2.16/-130 34917 [34787]	35304 -266 35038 2.2/-124 34914 2.07/-60 34854 [34794]	35272 -242 35030 2.1/-118 34912 [34794]		32709 -94 32614 1.81/-52 32562 [32510] [32509]	32735 -119 32617 2.19/-54 32563 [32509]	32736 -121 32615 2.28/-53 32562 2.12/-25 32537 [32512]	32713 -104 32609 2.17/-48 32561 [32513]		31388 -69 31319 2.09/-33 31286 [31253] [31127]	31424 -98 31326 2.51/-39 31287 [31248]	31450 -119 31331 2.77/-43 31288 2.53/-17 31271 [31254]	31440 -111 31329 2.71/-41 31288 [31247]	
	34216 -228 33988 2.2/-103 33885 [33782]	34190 -212 33978 2.2/-76 33902 2.09/-46 33856 [33790]	34151 -182 33969 2.05/-89 33880 [33771]		31973 -36 31937 1.49/-26 31911 [31885] [31825]	32006 -54 31952 2.12/-30 31922 [illegible]	32012 -71 31941 2.44/-29 31912 2.07/-14 31898 [31884]	31989 -53 31936 2.04/-26 31910 [31884]		31106 -14 31092 1.56/-9 31083 [31074] [31074]	31131 -34 31097 2.42/-13 31084 [31071]	31153 -52 31101 3.25/-16 31085 2.39/-7 31078 [31071]	31150 -49 31101 2.84/-17 31084 [31067]	
	32883 -129 32754 2.22/-58 32696 [32638]	32862 -117 32745 2.29/-51 32694 2.13/-24 32670 [32646]	32806 -73 32733 1.74/-42 32691 [32649]		31306 26 31332 0/-1 31331 [31330] [31326]	31345 -8 31337 1.13/-5 31332 [31326]	31363 -23 31340 2.88/-8 31332 2.67/-3 31329 [31329]	31350 -23 31327 2.6/-5 31322 [31327]		31006 1 31007 0/-1 31006 [31006] [31007]	31012 -4 31008 4.0/2[illegible] 31007 [31006]	31020 -10 31010 3.13/-3 31007 3.0/-1 31006 [31005]	31021 -11 31010 3.62/-3 31008 [31006]	
	2	2	2		2	2	2	2		2	2	2	2	

2

TABLE V

SOLUTIONS OF DIFFERENCE EQUATIONS FOR NDL, OPTION II (λ = ½)

(All values multiplied by 10^6.)

Row	N	ENIAC	FD(½)	CN(½)		ENIAC	FD(½)	CN(½)		ENIAC	FD(½)	CN(½)
		x/h = 0					x/h = ¼				x/h = ½	
20	8 16 32	40852	40852 -30 40022 3.3/-9 40813 [40810]				36560 -27 36533 3.85/-7 36526 [36524]	36547			33845 -24 33821 2.66/-9 33812 [33808]	
16	8 16 32	39973 -29 39944	39973 -32 39941 4.0/-8 39933 [39930]	39948		35730 -17 35713	35730 -24 35706 4.0/-6 35700 [35698]	35714		33177 16 33193	33177 -11 33166 2.6/-4 33162 [33160]	33192
12	8 16 32	38987 -34 38953	38987 -35 38952 4.4/-8 38944 [38941]	38959		34826 -21 34805	34826 -22 34804 3.67/-6 34798 [34796]	34808		32516 * 9 32525	32516 -1 32515 1.0/-1 32514 [32514]	32533
8	8 16 32	37838 -37 37801	37838 -37 37801 4.1/-9 37792 [37789]	37809		33817 -19 33798	33817 -19 33798 3.8/-5 33793 [33791]	33801		31876 9 31885	31876 7 31883 3.5/2 31885 [31886]	31898
4	8 16 32	36414 -60 36374	36414 -40 36374 4.0/-10 36364 [36361]	36387		32662 -12 32650	32662 -12 32650 6.0/-2 32648 [32647]	32654		31306 6 31322	31306 16 31322 5.0/3 31325 [31326]	31337
	[illegible]		4				4				2	

* - see Section 5

TABLE V

[illegible]S OF DIFFERENCE EQUATIONS FOR NDL, OPTION II (λ = ½)

(All values multiplied by 10^6.)

	x/λ = ¼				x/λ = ½				x/λ = 1		
ENIAC	FD(½)	CN(½)		ENIAC	FD(½)	CN(½)		ENIAC	FD(½)	CN(½)	
	36560 -27 36533 3.85/-7 36526 [36524]	36547			33845 -24 33821 2.66/-9 33812 [33808]				32211 -146 32065 2.2/-66 31999 [31933]		
35730 -17 35713	35730 -24 35706 4.0/-6 35700 [35698]	35714		33177 16 33193	33177 -11 33166 2.8/-4 33162 [33160]	33192		31734 118 31852	31734 -97 31637 2.2/-44 31593 [31549]	31785	
34826 -21 34805	34826 -22 34804 3.67/-6 34798 [34796]	34808		32516 9 32525	32516 -1 32515 1.0/-1 32514 [3251[illegible]]	32533		31352 79 31431	31352 -50 31302 2.1/-24 31278 [31254]	31402	
33817 -19 33798	33817 -19 33798 3.8/-5 33793 [33791]	33801		31876 9 31885	31876 7 31883 3.5/2 31885 [31886]	31898		31101 41 31142	31101 -12 31089 1.5/-8 31081 [31073]	31140	
2662 -12 2650	32662 -12 32650 6.0/-2 32648 [32647]	32654		31306 6 31322	31306 16 31322 5.0/3 31325 [31328]	31337		31006 8 31014	31006 -1 31007 •/1 31006 [31006]	31019	
	4				3				2		

2

TABLE VI

SOLUTIONS OF DIFFERENCE EQUATIONS FOR EDL, OPTION III

(All values multiplied by 10^6.)

		x = 0			x = 1/4			x = 1/2			x = 1	
Row	N	FD(½)	CWN(I)*		FD(½)	CWN(I)		FD(½)	CWN(I)		FD(½)	CWN(I)
	8	40822	40781		36521	36508		33820	33834		32198	32244
		13	23		8	13		-2	-16		-125	-170
20	16	40815	40804		36529	36521		33818	33818		32063	32074
		*/-2	4.6/5		*/-3	6.5/7		*/-7	2.3/-7		2.0/-64	2.3/-73
	32	40813	40809		36526	36523		33811	33811		31999	32001
			5.0/1			*/0			1.1/-4			2.3/-32
	64		40810			36523			33807			31969
			[40810]			[36523]			[33805]			[31937]
	8	39917	39893		35690	35674		33155	33170		31726	31776
		17	28		11	18		8	-6		-90	-129
16	15	39934	39921		35701	35692		33163	33164		31636	31647
		*/-2	4.0/7		*/-2	4.5/4		*/-2	2.0/-3		2.3/-43	2.5/-52
	32	39932	39928		35699	35696		33161	33161		31593	31595
			3.5/2			4.0/1			3.[illegible]-1			2.2/-23
	64		39930			35697			33160			31572
			[39931]			[35697]			[33160]			[31549]
	8	38924	38896		34766	34768		32498	32515		31348	31397
		20	34		13	11		15	-1		-46	-84
12	16	38944	38930		34779	34789		32513	32514		31302	31313
		*/-1	4.25/8		*/-3	1.8/6		*/1	*/0		1.7/-24	2.4/-32
	32	38943	38938		34796	34795		32514	32514		31278	31281
			4.0/2			6.0/1			*/0			2.3/-14
			38940			34796			32514			31267
			[38941]			[34796]			[32514]			[31253]
	8	37766	37737		33752	33765		31866	31886		31101	31139
		26	38		11	19		16	1		-12	-42
8	16	37792	37775		33773	33784		31882	31885		31089	31097
		*/-2	3.8/10		1.2/9	3.8/5		5.3/3	*/0		1.5/-8	3.0/-14
	32	37790	37785		33792	33789		31885	31885		31081	31083
			3.3/3			2.5/2			*/0			2.8/-5
	64		37788			33791			31885			31078
			[37789]			[33792]			[31885]			[31077]
	8	36332	36305		32642	32631		31306	31334		31006	31019
		31	3.8/42		5	11		16	-6		1	-9
4	16	36363	36347		32647	32642		31322	31328		31007	31010
		*/0	2.6/11		*/0	2.8/4		5.3/3	6.0/-1		*/-1	3.0/-3
	32	36363	36358		32647	32646		31325	31327		31006	31007
			5.5/2			*/0			*/0			3.0/-1
	64		36360			32646			31327			31006
			[36361]			[32646]			[31327]			[31005]
ρ			4			4			3			2

* - see Section 5

TABLE VII

SOLUTIONS OF DIFFERENCE EQUATIONS FOR [illegible], SERIES 1.
(All values multiplied by 10^5.)

		x/h = 0					x/h = ¼					x/h = ½					
Re	N	FD(r=½)	FD(r=¼)	CNM(I) *	CNM(II)*	CN(I)*	FD(r=½)	FD(r=¼)	CNM(I)	CNM(II)	CN(I)	FD(r=½)	FD(r=¼)	CNM(I)	CNM(II)	CN(I)	FD(r=½)
20	8 16 32 64	43615	43853 2970 46823	43621 1211 44632 0.37/3283 47915 B.L.	43025 822 43847 0.48/1730 45577	44053 B.L.**	38185	38316 146 38462	38131 49 38180 0.27/181 38361 B.L.	37909 60 37969 0.34/177 38146	38416 B.L.	34466	34503 -149 34354	34480 -156 34324 .44/-64 34260 B.L.	34412 -126 34286 .63/-48 34238	34534 B.L.	32412
16	8 16 32 64	41553 -162 41391 2.3/-71 41320 [41249]	41595 -194 41401 [illegible]	41486 -183 41303 4.2/-44 41259 .67/-12 41247 [41235]	41360 -90 41270 .27/-71 41199 [41128]	41586 -191 41395 [41204]	36660 -214 36446 1.8/-119 36327 [36208]	36696 -244 36452 [36208]	36633 -223 36410 .19/-105 36305 .06/-51 36254 [36203]	36546 -180 36366 2.14/-84 36282 [36198]	36690 -242 36448 [36208]	33546 -120 33426 1.76/-68 33358 [33290]	33581 -150 33431 [33281]	33574 -152 33422 .38/-68 33354 .13/-32 33322 [33290]	33535 -128 33407 .17/-59 33348 [33289]	33589 -158 33431 [33273]	31842 -145 31697 .20/-66 31631 [31565]
12	8 16 32 64	40160 -394 39766 .04/-193 39573 [39380]	40176 -410 39766 [39356]	40117 -388 39729 .72/-175 39554 .36/-74 39470 [39396]	40048 -354 39694 2.75/-157 39537 [39380]	40149 -375 39754 [39359]	35449 -239 35210 1.98/-121 35089 [34968]	35469 -257 35212 [34955]	35430 -239 35191 .15/-111 35080 .02/-55 35025 [34970]	35376 -205 35171 2.05/-100 35071 [34971]	35449 -246 35203 [34957]	32742 -86 32656 .75/-49 32607 [32558]	32772 -111 32661 [32550]	32771 -115 32656 .30/-50 32606 .08/-24 32582 [32558]	32741 -96 32647 2.18/-44 32603 [32559]	32777 -118 32659 [32541]	31395 -68 31327 2.06/-33 31294 [31261]
8	8 16 32 64	38823 -459 38364 .1/-220 38144 [37924]	38833 -471 38362 [37891]	38788 -449 38339 2.18/-206 38133 .08/-99 38034 [37935]	38734 -415 38319 2.14/-194 38125 [37931]	38779 -452 38327 [37875]	34237 -199 34038 2.0/-100 33938 [33838]	34257 -219 34038 [33819]	34227 -202 34025 .17/-93 33932 .11/-44 33888 [33844]	34181 -170 34011 .05/-83 33928 [33845]	34234 -205 34029 [33824]	31982 -33 31949 .38/-25 31924 [31899]	32016 -64 31952 [31892]	32022 -69 31953 .37/-29 31924 .23/-13 31911 [31890]	31998 -51 31947 .04/-25 31922 [31897]	32024 -70 31954 [31894]	31109 -13 31096 1.3/-10 31086 [31076]
4	8 16 32 64	37261 -456 36805 2.14/-213 36592 [36379]	37277 -473 36804 [36331]	37232 -449 36783 .75/-200 36583 .13/-94 36489 [36375]	37171 -405 36766 2.15/-188 36578 [36390]	37236 -450 36786 [36336]	32850 -89 32761 .65/-54 32707 [32653]	32890 -126 32764 [32638]	32869 -115 32754 .75/-51 32703 .72/-23 32680 [32657]	32811 -70 32741 1.71/-41 32700 [32659]	32871 -116 32755 [32639]	31308 36 31344 0/0 31344 [31334]*	31348 -8 31340 [31332]	31366 -23 31343 2.84/-8 31335 2.67/-3 31332 [31329]	31353 -14 31339 2.8/-5 31334 [31329]	31364 -23 31343 [31320]	31008 1 31009 0/-1 31008 [31008]*
	ε	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

* - see Section 5
** - overflow

TABLE VII

SOLUTIONS OF DIFFERENCE EQUATIONS FOR [illegible], OPTION 1.
(All values multiplied by 10^5.)

x/h = ¼				x/h = ½					x/h = 1				
FD(r=½)	CNM(I)	CNM(II)	CN(I)	FD(r=½)	FD(r=¼)	CNM(I)	CNM(II)	CN(I)	FD(r=½)	FD(r=¼)	CNM(I)	CNM(II)	CN(I)
38316 146 38462	38131 49 38180 0.27/181 38361 B.L.	37908 60 37969 0.34/177 38146	38416 B.L.	34466	34503 -149 34354	34480 -156 34324 2.44/-64 34260 B.L.	34412 -126 34286 2.63/-48 34238	34534 B.L.	32419	32458 -263 32195	32478 -281 32197 2.51/-112 32085 B.L.	32452 -262 32190 2.43/-108 32082	32487 B.L.
36696 -244 36452 [36208]	36633 -223 36410 2.12/-105 36305 2.06/-51 36254 [36203]	36546 -180 36366 2.14/-84 36282 [36198]	36690 -242 36448 [36208]	33546 -120 33426 1.76/-68 33358 [33290]	33581 -150 33431 [33281]	33574 -152 33422 2.38/-68 33354 2.13/-32 33322 [33290]	33535 -128 33407 2.17/-59 33348 [33289]	33589 -158 33431 [33273]	31842 -145 31697 2.20/-66 31631 [31565]	31884 -179 31705 [31526]	31908 -199 31709 2.58/-78 31631 2.30/-33 31600 [31567]	31890 -185 31705 2.53/-73 31632 [31559]	31911 -200 31711 [31511]
35469 -257 35212 [34955]	35430 -239 35191 2.15/-111 35080 2.02/-55 35025 [34970]	35376 -205 35171 2.05/-100 35071 [34971]	35419 -216 35203 [34987]	32742 -86 32656 1.75/-49 32607 [32558]	32772 -111 32661 [32550]	32771 -115 32656 2.30/-50 32606 2.08/-24 32582 [32558]	32743 -96 32647 2.18/-44 32603 [32559]	32777 -118 32659 [32541]	31395 -68 31327 2.06/-33 31294 [31261]	31433 -99 31334 [31235]	31459 -120 31339 2.86/-42 31297 2.33/-18 31279 [31261]	31448 -111 31337 2.72/-41 31296 [31255]	31460 -20 31340 [31320]
34257 -219 34038 [33819]	34227 -202 34025 2.17/-93 33932 2.11/-44 33888 [33844]	34181 -170 34011 2.05/-83 33928 [33845]	34234 -205 34029 [33824]	31982 -33 31949 1.32/-25 31924 [31899]	32016 -62 31954 [31892]	32022 -69 31953 2.37/-29 31924 2.23/-13 31911 [31898]	31998 -51 31947 2.04/-25 31922 [31897]	32024 -70 31954 [31884]	31109 -13 31096 1.3/-10 31086 [31076]	31135 -35 31100 [31065]	31157 -52 31105 3.06/-17 31088 3.83/-6 31082 [31076]	31154 -50 31104 3.13/-16 31088 [31072]	31157 -52 31105 [31053]
32890 -126 32764 [32638]	32869 -115 32754 2.75/-51 32703 2.72/-23 32680 [32657]	32811 -70 32741 1.71/-41 32700 [32659]	32871 -116 32755 [32639]	31308 26 31334 0/0 31334 [31334]	31348 -8 31340 [31332]	31366 -23 31343 2.09/-8 31335 2.67/-3 31332 [31329]	31353 -14 31339 2.8/-5 31334 [31329]	31366 -23 31343 [31320]	31008 1 31009 0/-1 31008 [31008]	31014 -4 31010 [31006]	31021 -10 31011 5.0/-2 31009 2.0/-2 31008 [31007]	31021 -11 31010 3.67/-3 31009 [31006]	31021 -10 31011 [31008]
2	2	3	3	3	[illegible]	2	2	2	2	2	2	2	2

2

TABLE VIII

SOLUTIONS OF DIFFERENCE EQUATIONS FOR NDN, OPTION II ($r = \frac{1}{2}$).

(All values multiplied by 10^6.)

		$x/L = 0$			$x/L = \frac{1}{4}$			$x/L = \frac{1}{2}$			$x/L = \frac{3}{4}$		
Row	N	ENIAC	FD($\frac{1}{2}$)	CN($\frac{1}{2}$)	ENIAC	FD($\frac{1}{2}$)	CN($\frac{1}{2}$)	ENIAC	FD($\frac{1}{2}$)	CN($\frac{1}{2}$)	ENIAC	FD($\frac{1}{2}$)	CN($\frac{1}{2}$)
	8		B.I.**	B.I.		B.I.	B.I.		B.I.	B.I.		B.I.	B.I.
20	16		B.I.	B.I.		B.I.	B.I.		B.I.	B.I.		B.I.	B.I.
	32		B.I.			B.I.			B.I.			B.I.	
	64												
	8	41996	41995	42061	36287	36287	36365	33308	33308	33355	31757	31757	31817
		-500	-564	-610	-51	-58		20	-8	-46	124	-98	-146
16	16	41436	41432	41451	36236	36229		33328	33300	33309	31881	31659	31673
		3.57/-146	3.94/-143		3.64/-14	3.05/-19		1.82/11	1.6/-5		1.91/65	2.18/-45	
	32	41296	41289		36222	36210		33339	33295		31946	31614	
		4.24/-33			4.67/-3			1.83/6			1.91/34		
	64	41263			36219			33345			31980		
		[41254]	[41241]		[36218]	[36204]		[33351]	[33290]		[32084]	[31569]	
	8	39610	39610	39596	35013	35013	35017	32559	32559	32587	31361	31361	31434
		-142	-143	-155	-28	-29		11	2	-21	81	-50	-91
12	16	39468	39467	39441	34985	34984		32570	32561	32566	31442	31311	31343
		3.4/-45	3.42/-45		4.0/-7	3.22/-9		1.83/6	2.0/1		1.98/41	2.08/-48	
	32	39403	39402		34978	34975		32576	32560		31483	31287	
		3.75/-12			3.5/-2			2.0/3			1.86/22		
	64	39391			34976			32579			31505		
		[39387]	[39387]		[34975]	[34973]		[32582]	[32561]		[31527]	[31263]	
	8	38043	38043	38018	33872	33872	33864	31887	31887	31913	31105	31105	31145
		-77	-77	-62	-20	-20		11	9	-11	41	-12	-44
8	16	37966	37966	37956	33852	33852		31898	31896	31902	31146	31093	31101
		3.67/-21	3.67/-21		4.0/-5	4.0/-5		1.57/7	4.5/2		2.05/20	1.5/-8	
	32	37945	37945		33847	33847		31905	31898		31166	31085	
		3.5/-6			5.0/-1			7.0/1			2.0/10		
	64	37939			33846			31906			31176		
		[37937]	[37938]		[33846] *	[33845]		[31903]	[31900]		[31186]	[31077]	
	8	36461	36464	36439	32671	32671	32666	31308	31308	31340	31009	31008	31021
		-46	-49	-35	-10	-10		17	17	-8	7	1	-10
4	16	36415	36415	36404	32661	32661		31325	31325	31332	31016	31009	31011
		3.77/-13	3.77/-13		3.33/-3	3.33/-3		5.67/3	5.67/3		1.75/4	*/-1	
	32	36402	36402		32658	32658		31328	31328		31020	31008	
		4.33/-3			*/0			3.0/1			2.0/2		
	64	36399			32658			31329			31022		
		[36398]	[36398]		[32658] *	[32657]		[31330]	[31331]		[31024]	[31008] *	
	q	4	4		4	4		2	2		2	2	

* = see Section 5
** = overflow

TABLE VIII

SOLUTIONS OF DIFFERENCE EQUATIONS FOR NDS, OPTION II (r = ½).

(All values multiplied by 10^6.)

		x/A = 0				x/A = ¼			
Row	N	ENIAC	FD(½)	CN(½)		ENIAC	FD(½)	CN(½)	
	8		B.L.**	B.L.			B.L.	B.L.	
20	16		B.L.	B.L.			B.L.	B.L.	
	32		B.L.				B.L.		
	64								
	8	41996	41995	42061		36287	36287	36355	
16	16	-500 41436	-564 41432	-610 41451		-51 36236	-58 36229		
	32	3.57/-140 41296	3.94/-143 41289			3.64/-14 36222	3.05/-19 36210		
	64	4.24/-33 41263				4.67/-3 36219			
		[41254]	[41241]			[36218]	[36204]		
	8	39610	39610	39596		35013	35013	35017	
12	16	-162 39448	-163 39447	-155 39441		-28 34985	-29 34984		
	32	3.6/-45 39403	3.62/-45 39402			4.0/-7 34978	3.22/-9 34975		
	64	3.75/-12 39391				3.5/-2 34976			
		[39387]	[39387]			[34975]	[34973]		
	8	38043	38043	38018		33872	33872	33864	
8	16	-77 37966	-77 37966	-62 37956		-20 33852	-20 33852		
	32	3.67/-21 37945	3.67/-21 37945			4.0/-5 33847	4.0/-5 33847		
	64	3.5/-6 37939				5.0/-1 33846			
		[37937]	[37938]			[33846] *	[33845]		
	8	36464	36464	36439		32671	32671	32666	
4	16	-49 36415	-49 36415	-35 36404		-10 32661	-10 32661		
	32	3.77/-13 36402	3.77/-13 36402			3.33/-3 32658	3.33/-3 32658		
	64	4.33/-3 36399				*/0 32658			
		[36398]	[36398]			[32658] *	[32657]		
	q	4	4			4	4		
		* = see Section 5 ** = overflow							

		x/A = ½				x/A = ¾			
Row	N	ENIAC	FD(½)	CN(½)		ENIAC	FD(½)	CN(½)	
	8		B.L.	B.L.			B.L.	B.L.	
20	16		B.L.	B.L.			B.L.	B.L.	
	32		B.L.				B.L.		
	64								
	8	33308	33308	33355		31757	31757	31817	
16	16	20 33328	-8 33300	-46 33309		124 31881	-98 31659	-144 31673	
	32	1.82/11 33339	1.6/-5 33295			1.91/65 31946	2.18/-45 31614		
	64	1.83/6 33345				1.91/34 31980			
		[33351]	[33290]			[32024]	[31569]		
	8	32559	32559	32587		31361	31361	31414	
12	16	11 32570	2 32561	-21 32566		81 31442	-50 31311	-91 31323	
	32	1.83/6 32576	2.0/1 32560			1.98/41 31483	2.08/-24 31287		
	64	2.0/3 32579				1.86/22 31505			
		[32580]	[32561]			[31517]	[31263]		
	8	31887	31887	31913		31105	31105	31145	
8	16	11 31898	9 31896	-11 31902		41 31146	-12 31093	-44 31101	
	32	1.57/7 31901	4.5/2 31898			2.05/20 31166	1.5/-8 31085		
	64	7.0/1 31902				2.0/10 31186			
		[31903]	[31900]			[31186]	[31077]		
	8	31308	31308	31340		31009	31008	31021	
4	16	17 31325	17 31325	-8 31332		7 31016	1 31009	-10 31011	
	32	5.67/3 31328	5.67/3 31328			1.75/4 31020	*/-1 31008		
	64	3.0/1 31329				2.0/2 31022			
		[31330]	[31331]			[31024]	[31008] *		
	q	2	2			2	2		

Appendix

Formulas Used for Ordvac Program

Notation:

$u_{i,j}$ - as discussed in Section 3

$u_{i,j}^{(n)}$ - value of function for n-th iteration

Bracketed term included only for the non-linear case.

FD - BW

$$u_{i,j+1} = r(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + u_{i,j} + \left[k\gamma e^{u_{i,j}}\right]; \quad 1 \le i \le M-1; \; j \ge 0$$

$$\left.\begin{array}{l} u_{0,j+1} = \phi_{j+1} \\ u_{M,j+1} = \psi_{j+1} \end{array}\right\} \; j \ge 0$$

FD - ND

$$u_{i,j+1} = r(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + u_{i,j} + \left[ke^{-1/u_{i,j}}\right]; \quad 1 \le i \le M-1; \; j \ge 0$$

Op I $\quad u_{0,j+1} = \dfrac{Hhu_g + u_{1,j+1}}{1 + Hh} \qquad j \ge 0$

Op II $\quad u_{0,j+1} = u_{1,j} + Hh(u_g - u_{0,j}) + \left[ke^{-1/u_{0,j}}\right] \qquad j \ge 0$

$r = \frac{1}{2}$ only

Op III' $\quad u_{0,j+1} = \dfrac{u_{1,j+1} + Hhu_g + \frac{1}{2r}\left\{u_{0,j} + \left[ke^{-1/u_{0,j}}\right]\right\}}{1 + Hh + \frac{1}{2r}} \qquad j \ge 0$

All Options:

$$u_{M,j+1} = u_{M-1,j+1}$$

CN - BV

$$u^{(0)}_{i,j+1} = u_{i,j+1} \quad \text{by FD method} \quad \text{if } r \le \tfrac{1}{2}$$

$$u^{(0)}_{i,j+1} = u_{i,j} \qquad \text{if } r > \tfrac{1}{2}$$

$$1 \le i \le M-1 \; ; \; j \ge 0$$

$$u^{(n)}_{i,j+1} = (\omega-1)(\bar{u}^{(n)}_{i,j+1} - u^{(n-1)}_{i,j+1}) + \bar{u}^{(n)}_{i,j+1}$$

where:

$$\bar{u}^{(n)}_{i,j+1} = \frac{r}{2+2r}(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + \frac{1}{1+r} u_{i,j} + \frac{r}{2+2r}(u^{*}_{i-1,j+1} + u^{(n-1)}_{i+1,j+1}) + \left[\frac{1}{1+r} k \gamma e^{\left(\frac{u_{i,j} + u^{(n-1)}_{i,j+1}}{2}\right)}\right]$$

where:

$$u^{*}_{i-1,j+1} = u^{(n-1)}_{0,j+1} \qquad i = 1 \; ; \; j \ge 0$$

$$u^{*}_{i-1,j+1} = u^{(n)}_{i-1,j+1} \qquad 2 \le i \le M-1 \; ; \; j \ge 0$$

$$u^{(n)}_{0,j+1} = \phi_{j+1}$$

$$n \ge 0 \; ; \; j \ge 0$$

$$u^{(n)}_{M,j+1} = \psi_{j+1}$$

CN – ND

$$u^{(0)}_{i,j+1} = u_{i,j+1} \text{ by FD method if } r \leq \tfrac{1}{2}$$

$$u^{(0)}_{i,j+1} = u_{i,j} \qquad \text{if } r > \tfrac{1}{2}$$

$$0 \leq i \leq M \,;\quad j \geq 0$$

$$u^{(n)}_{i,j+1} = (\omega-1)\left(\bar{u}^{(n)}_{i,j+1} - u^{(n-1)}_{i,j+1}\right) + \bar{u}^{(n)}_{i,j+1}$$

where:

$$\bar{u}^{(n)}_{i,j+1} = \frac{r}{2+2r}(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + \frac{1}{1+r}u_{i,j} + \frac{r}{2+2r}\left(u^{*}_{i-1,j+1} + u^{(n-1)}_{i+1,j+1}\right) + \left[\frac{1}{1+r}\,k\,\exp\left\{-\frac{1}{\left(\frac{u_{i,j} + u^{(n-1)}_{i,j+1}}{2}\right)}\right\}\right]$$

where:

$$u^{*}_{i-1,j+1} = u^{(n-1)}_{0,j+1} \qquad i = 1\,;\quad j \geq 0$$

$$u^{*}_{i-1,j+1} = u^{(n)}_{i-1,j+1} \qquad 2 \leq i \leq M-1\,;\quad j \geq 0$$

Op I
$$u^{(n)}_{0,j+1} = \frac{Hhu_g + u^{(n)}_{1,j+1}}{1 + Hh} \qquad n > 0\,;\ j \geq 0$$

Op II
$$u^{(n)}_{0,j+1} = u_{1,j} + Hh(u_g - u_{0,j}) + \left[ke^{-1/u_{0,j}}\right] \qquad n > 0\,;\ j \geq 0$$

$r = \tfrac{1}{2}$ only

Op III
$$u^{(n)}_{0,j+1} = \frac{u^{(n)}_{1,j+1} + Hhu_g + \frac{1}{2r}\left\{u_{0,j} + \left[ke^{-1/u^{(n-1)}_{0,j+1}}\right]\right\}}{1 + Hh + 1/2r} \qquad n > 0;\ j \geq 0$$

All Options: $u^{(n)}_{M,j+1} = u^{(n)}_{M-1,j+1}$

CNM - BV

$$d_0 = 0$$

$$d_1 = \frac{r}{2+2r}(u_{0,j} + u_{2,j} - 2u_{1,j}) + \frac{1}{1+r}u_{1,j} + \frac{r}{2+2r}u_{0,j+1} + \left[\frac{1}{1+r}k\gamma e^{u_{1,j}}\right] \quad j \geq 0$$

$$d_i = \frac{r}{2+2r}(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + \frac{1}{1+r}u_{i,j} + \left[\frac{1}{1+r}k\gamma e^{u_{i,j}}\right] \quad 2 \leq i \leq M-2;\ j \geq 0$$

$$d_{M-1} = \frac{r}{2+2r}(u_{M-2,j} + u_{M,j} - 2u_{M-1,j}) + \frac{1}{1+r}u_{M-1,j} + \frac{r}{2+2r}u_{M,j+1} + \left[\frac{1}{1+r}k\gamma e^{u_{M-1,j}}\right] \quad j \geq 0$$

$$d_M = 0$$

$$b_0 = 0$$

$$b_1 = -\frac{r}{2+2r}$$

$$b_i = \frac{-r/2+2r}{1 + \left(\frac{r}{2+2r}\right)b_{i-1}} \quad 2 \leq i \leq M-2$$

$$b_{M-1} = 0$$

$$q_0 = 0$$

$$q_1 = d_1$$

$$q_i = \frac{d_i + \left(\frac{r}{2+2r}\right)q_{i-1}}{1 + \left(\frac{r}{2+2r}\right)b_{i-1}} \quad 2 \leq i \leq M-1$$

$$u_{M-1,j+1} = q_{M-1}$$

$$u_{i,j+1} = q_i - b_i u_{i+1,j+1} \quad 1 \leq i \leq M-2$$

CNM - ND Op I

$$d_0 = Hhu_g$$

$$d_i = \frac{r}{2+2r}\left(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}\right) + \frac{\lambda}{1+r} u_{i,j} + \left[\frac{\lambda}{1+r} ke^{-1/u_{i,j}}\right] \qquad 1 \leq i \leq M-1$$

$$d_M = 0$$

$$b_0 = \frac{-1}{1+Hh}$$

$$b_i = \frac{-r/2+2r}{1+\left(\frac{r}{2+2r}\right)b_{i-1}} \qquad 1 \leq i \leq M-1$$

$$b_M = 0$$

$$q_0 = \frac{d_0}{1+Hh}$$

$$q_i = \frac{d_i + \left(\frac{r}{2+2r}\right)q_{i-1}}{1 + \left(\frac{r}{2+2r}\right)b_{i-1}} \qquad 1 \leq i \leq M-1$$

$$q_M = \frac{q_{M-1}}{1+b_{M-1}}$$

$$u_{M,j+1} = q_M$$

$$u_{i,j+1} = q_i - b_i u_{i+1,j+1} \qquad 0 \leq i \leq M-1$$

CNN - ND Op II $\quad r = \frac{1}{2}$ only

$$d_0 = 0$$

$$d_1 = \frac{r}{2+2r}(u_{0,j} + u_{2,j} - 2u_{1,j}) + \frac{1}{1+r}u_{1,j} + \frac{r}{2+2r}u_{0,j+1} + \left[\frac{1}{1+r}ke^{-1/u_{1,j}}\right]$$

$$d_i = \frac{r}{2+2r}(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + \frac{1}{1+r}u_{i,j} + \left[\frac{1}{1+r}\ ke^{-1/u_{i,j}}\right] \qquad 2 \leq i \leq M-1$$

$$d_M = 0$$

$$b_0 = 0$$

$$b_1 = -\frac{r}{2+2r}$$

$$b_i = \frac{-r/2+2r}{1+\left(\frac{r}{2+2r}\right)b_{i-1}} \qquad 2 \leq i \leq M-1$$

$$b_M = 0$$

$$q_0 = 0$$

$$q_1 = d_1$$

$$q_i = \frac{d_i + \left(\frac{r}{2+2r}\right)q_{i-1}}{1 + \left(\frac{r}{2+2r}\right)b_{i-1}} \qquad 2 \leq i \leq M-1$$

$$q_M = \frac{q_{M-1}}{1+b_{M-1}}$$

$$u_{M,j+1} = q_M$$

$$u_{i,j+1} = q_i - b_iu_{i+1,j+1} \qquad 1 \leq i \leq M-1$$

CNN - ND Op III'

$$d_0 = Hhu_g + \frac{1}{2r}\left\{u_{0,j} + \left[ke^{-1/u_{0,j}}\right]\right\}$$

$$d_i = \frac{r}{2+2r}(u_{i-1,j} + u_{i+1,j} - 2u_{i,j}) + \frac{1}{1+r}u_{i,j} + \left[\frac{1}{1+r}ke^{-1/u_{i,j}}\right]$$

$1 \leq i \leq M-1$

$$d_M = 0$$

$$b_0 = \frac{-1}{1+Hh+1/2r}$$

$$b_i = \frac{-r/2+2r}{1+\left(\frac{r}{2+2r}\right)b_{i-1}} \qquad 1 \leq i \leq M-1$$

$$b_M = 0$$

$$q_0 = \frac{d_0}{1+Hh+1/2r}$$

$$q_i = \frac{d_i + \left(\frac{r}{2+2r}\right)q_{i-1}}{1+\left(\frac{r}{2+2r}\right)b_{i-1}} \qquad 1 \leq i \leq M-1$$

$$q_M = \frac{q_{M-1}}{1+b_{M-1}}$$

$$u_{M,j+1} = q_M$$

$$u_{i,j+1} = q_i - b_i u_{i+1,j+1} \qquad 0 \leq i \leq M-1$$

DFF

Compute 2^{α} rows by FD technique using r', where $r = r' \cdot 2^{\alpha}$; $0 < r' \leq \frac{1}{2}$.
Let 2^{α} th row by FD be $j = 1$ (first row) for DFF. Then:

$$u_{i,j+1} = \frac{2r}{1+2r}\left(u_{i-1,j} + u_{i+1,j} - 2u_{i,j-1}\right) + u_{i,j-1} + \left[\frac{2}{1+2r} k e^{\phi}\right]$$

$$1 \leq i \leq M-1 \; ; \quad j \geq 1$$

$$\text{where } e^{\phi} = \begin{cases} \gamma e^{u_{i,j}} & \text{BV} \\ e^{-1/u_{i,j}} & \text{ND} \end{cases}$$

The boundary conditions are the same as for the FD method, both BV and ND.

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